KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLII

No. 4

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

Embryology. — *On lens-induction.* By M. W. WOERDEMAN.

(Communicated at the meeting of March 25, 1939.)

On a previous occasion I communicated the results of an investigation into the inducing capacity of the eye vesicle in *Axolotl* embryos (see Proc. Kon. Ned. Akad. v. Wetensch., Vol. XLI, 4, 336, 1938).

Eye vesicles of different ages were grafted into the abdominal wall and covered with ectoderm, likewise of varying age. The eye vesicles proved to be able to induce not only lenses but also nasal pits and ear vesicles.

By this time I can communicate that experiments on embryos of *Triton taeniatus* yielded similar results.

A number of supplementary observations, however, made it necessary to pay special attention to the induction of lenses.

As a rule this is conceived as follows: During embryonic development the eye vesicle (presumptive retina) comes into contact with the ectoderm of the head and there induces a cell proliferation which is transformed into a vesicle (lens vesicle) and then separates from the ectoderm. The wall of the vesicle, in touch with the presumptive retina (I will call it "inner wall"), displays a differentiation of the cells into lens fibres, the wall turned towards the ectoderm ("outer wall") producing the lens epithelium. The formation of lens fibres is likewise based upon an inducing activity of the retina.

In my experimental embryos now I found not only lenses normally situated in the pupil of the grafted eye cup but also some which without contact with the eye lay rather far away from the graft or were in touch with the pigment layer of the retina (tapetum). Moreover, I observed embryos containing a large number of lenses in the neighbourhood of the grafted eye vesicle, some of which were more highly differentiated, others less so.

Frequently I found the grafted eyes in an abnormal position, e.g. with the pupil turned inside, while they had all been grafted with the pupil on the outside.

Further I often found a lens in the pupil, situated far from the ectoderm, which gave the impression that the eye vesicle had induced a lens out of other material than the epidermis of the head. As a matter of course, such abnormal positions occur frequently on grafting the eye vesicle into the blastocoele, which operation I performed a number of times as well (see the above-mentioned report in the Proceedings).

In the literature similar observations have been published, while conclusions were drawn which I consider not entirely proved.

In order to be able to judge these conclusions, I have considered how the lens rudiment will react when sooner or later in its development the contact with the eye vesicle is broken.

For various reasons the experiments were first made on embryos of *Rana esculenta.*

As starting-point were taken the stages with beginning tail bud formation. They were chosen in such a way that the lens anlage of the experimental embryo had not yet been formed into a vesicle. I might now have extirpated the eye rudiment and left the lens proliferation where it was, as some workers did, but in order to eliminate the influence of neighbouring head organs, which a priori may not be excluded (OKADA and MIKAMI, DRAGOMIROW 1929, FILATOW 1925, IKEDA, HOLTFRETER), I chose another technique. The lens proliferation was extirpated and grafted under the ectoderm of the abdomen. Lens proliferations were also cultivated in vitro, but this method yielded less satisfactory results.

Five or six days after the operation the experimental embryos were killed and examined microscopically.

It became evident that very young lens proliferations had not developed further than to the vesicular stage and showed no or hardly any lens fibre formation. Slightly older lens proliferations had developed into lens vesicles with fibres, but the fibre formation was neither regular nor extensive. In some cases fibres developed from the original inner wall of the lens vesicle, in spite of the fact that against the outer wall a fragment of retinal tissue was situated which had been grafted at the same time.

Consequently I obtained the impression that the formation of a vesicle and of fibres during the development of the lens may take place without contact with the presumptive retina, i.e. that these processes are induced already in an early stage of development and that only a short contact of eye vesicle and ectoderm is needed.

For a satisfactory development of the fibres, however, apparently a longer contact or a constant influence of the retina is necessary (see also PERRI, 1934).

This is in contradiction with the opinion of SPEMANN that in *Rana esculenta* a so-called "independent" lens formation occurs. During numerous previous experiments on lens development in this animal I never observed an independent lens formation. Other workers likewise state that they cannot confirm SPEMANN's opinion, although in the literature also observations are recorded involving a confirmation. It is not impossible that here differences between *esculenta* races are responsible for the varying results.

The opinion, that for induction of a lens only a short contact between epidermis and eye vesicle is required, is also supported by DRAGOMIROW (1930) and by FILATOW (1934). Concerning the exact time, at which the lens components are determinated, opinions still differ.

Reverting now to the observations which gave rise to the above-mentioned

investigation. I am inclined to think that an eye vesicle under ectoderm can induce numerous lenses, if by the growth of neighbouring organs it is placed in different positions and, now in this place, then in another, comes into contact with the ectoderm. My previous investigation, namely, revealed that the eye vesicle during a long time possesses the capacity to induce lenses. The lenses, induced in this way, will develop differently, according as the eye sooner or later loses contact with the lens anlage.

If during that process the eye vesicle is turned, it may take the induced lens along with it. The latter, if replaced as well, may later give the incorrect impression that it has been induced out of other material than ectoderm. Thus the results of POPOFF (1937) might be explained. He thought, namely, that he had found lens formation out of different tissues if an abnormally orientated eye vesicle came into contact with them. Only his recent researches (grafting of different tissues into the cavity of the eye cup) can produce evidence in favour of his opinion.

Besides, the fact that a lens is found against the tapetum of the eye (POLITZER) may not lead to the conclusion that this layer also can induce a lens.

Finally I will mention that in *Triton taeniatus* I repeatedly noticed lens formation out of the margin of the iris and even directly out of the retina of the grafted eye vesicles. Nevertheless, these eyes had also induced lenses out of ectoderm. The presence of a regenerated lens consequently does not deprive the eye of its capacity to induce lenses.

The phenomenon that an eye vesicle, grafted at a very young stage without a lens rudiment, regenerates a lens out of its retina or iris may also account for some cases described in the literature of so-called lens induction out of other tissues than ectoderm. The observed lenses might have been regenerated lenses. In *Axolotl* and *Rana esculenta* I observed these regenerated lenses only rarely, but there is no doubt that this type of lens formation occurs. In *Triton taeniatus* the phenomenon may be frequently observed.

LITERATURE.

DRAGOMIROW, N., ROUX' Arch. Entw. mech., **116**, 633 (1929).
———————, ROUX' Arch. Entw. mech., **123**, 206 (1930).
FILATOW, D., ROUX' Arch. Entw. mech., **104** and **105** (1925).
———————, Biol. Zh., **3**, 261 (1934).
HOLTFRETER, J., ROUX' Arch. Entw. mech., **134**, 466 (1936).
IKEDA, Y., Arb. anatom. Inst. Sendai, **21**, 1 (1938).
MANGOLD, O., Ergebn. d. Biol., **7**, 193 (1931) (containing an extensive bibliography).
OKADA, Y. K. and Y. MIKAMI, Proceed. imper. Acad. Tokyo, **13**, 7, 283 (1937).
PERRI, T., Atti Accad. naz. Lincei, VI s., **19**, 821 (1934).
POLITZER, G., ROUX' Arch. Entw. mech., **135**, 753 (1937).
POPOFF, W. W., Zool. Jahrb. Abt. allg. Zool. u. Physiol., **58**, 23 (1937).
SPEMANN, H., Zool. Jahrb., **32**, 1 (1912).
WOERDEMAN, M. W., Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 336 (1938).

Physics. — *Some considerations on the fields of stress connected with dislocations in a regular crystal lattice.* I. By J. M. BURGERS. (Mededeeling N⁰. 34 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of January 28, 1939.)

1. In order to explain the mechanism of plastic deformation of a crystal in its most simple form, as it is presented by the shearing process due to slipping along planes of a definite crystallographic orientation, several authors have assumed that the basic phenomenon leading to slip is the migration through the lattice of a well defined type of deviation from the ideal structure, a so-called *dislocation* [1]).

It has been in particular TAYLOR who has investigated the characteristic properties of an elementary, two-dimensional type of dislocation, the possibilities for its displacement through the lattice, and the influence of the fields of stress connected with a system of such dislocations upon this displacement [2]). An account of some of the results of this work, together with suggestions for certain modifications which made it possible to construct a connection with views developed by BECKER and by OROWAN, has been given by W. G. BURGERS and the present author in the "First Report on Viscosity and Plasticity", pp. 199 and seq. The problem, however, presented itself whether the two-dimensional type of dislocation, which must extend in a straight line through the lattice from one boundary surface of the crystal to the opposite boundary, really leads to an appropriate description of what is to be found in an actual crystal; it would appear that dislocations characterized by disturbances of a more general, three-dimensional type, which for instance may be confined to a region of finite extent, might lead to a more adequate picture [3]). It is the object of the following pages to make a few contributions towards the development

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1) Compare: "First Report on Viscosity and Plasticity" (Verhand. Kon. Nederl. Akad. v. Wetenschappen te Amsterdam, 1e sectie, XV, No. 3, 1935), p. 198 and the literature mentioned there; "Second Report on Viscosity and Plasticity" (ibidem, XVI, No. 4, 1938), p. 200.

See also papers by A. KOCHENDÖRFER, Zeitschr. f. Physik **108**, p. 244, 1938 and Zeitschr. f. Metallkunde **30**, p. 299, 1938.

2) G. I. TAYLOR, Proc. Roy. Soc. (London) **A 145**, p. 362, 1934.

3) "Second Report on Viscosity and Plasticity", p. 201. — KOCHENDÖRFER in his second paper (see footnote 1, above) alludes to the same problem; however, the few lines devoted by him to this matter (*l.c.* p. 300, second column) apparently are not based upon a sufficiently developed investigation of the geometric features of dislocations of three-dimensional type.

of such a picture, by investigating some of the geometrical relationships presented by dislocations of three-dimensional nature, and developing expressions for the fields of stress connected with them. It must be remarked that the treatment is of a preliminary character, and the reader will note several points where further work will be necessary.

2. *Introductory geometrical considerations on dislocations of various form.* — A schematical picture of a lattice with a two-dimensional "unit dislocation" of the type as considered by TAYLOR, is given in fig. 1. The

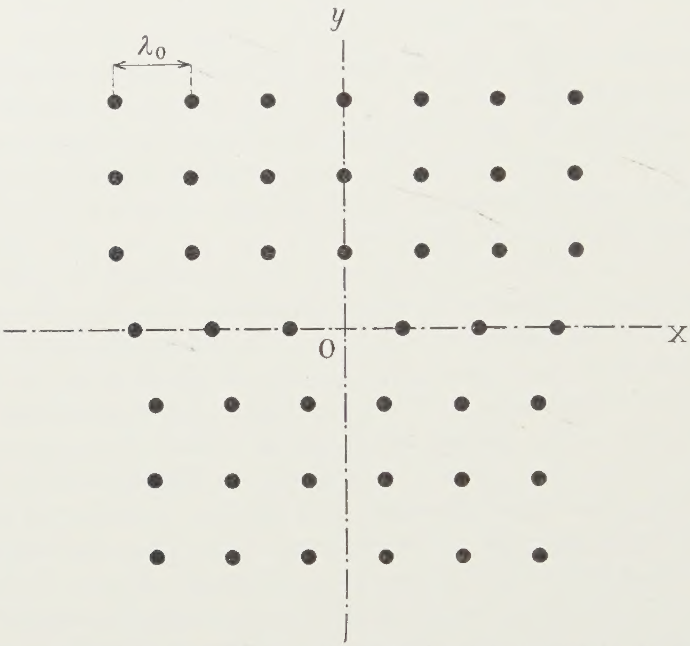

Fig. 1. Elementary type of a two-dimensional dislocation, having the z-axis (perpendicular to the plane of the paper) as its singular line.

disturbance presented by the lattice in this case can be described by stating that above a definite horizontal plane, say the x, z-plane, every row of atoms parallel to the x-axis contains one atom more than every row below this plane. The dislocation can be obtained by imagining the lattice to be cut along the upper half of the y, z-plane (*i.e.* the half plane $x = 0$, $y > 0$), and inserting an extra layer of atoms into this cut.

It will be evident that the deformations appearing in the lattice in consequence of this process (*i.e.* the deviations from their original cubical form, which are shown by the cells of the lattice) decrease indefinitely with increasing distances from the z-axis. Instead of the deformations of the cells we will consider the *displacements* of the atoms from their normal positions. When the components of the displacement are denoted by u, v, w, it will be seen that although these quantities in reality are defined only for the (enumerable) set of lattice points where atoms are to be found,

they can be considered as being determined by functions of the coordinates x, y, z, which in general are everywhere continuous and finite. It is only at the points of the z-axis and its immediate neighbourhood, that these functions lose their meaning; moreover, in the case of the function giving the values of u the following point is to be noted: When in the half plane $x=0$, $y<0$ we assume $u=0$, then in the region where y is positive we shall find that u approaches to the value $+\frac{1}{2}\lambda_0$ for $x>0$, and to the value $-\frac{1}{2}\lambda_0$ for $x<0$; hence the corresponding function will be discontinuous at the points of the half plane $x=0$, $y>0$. The explicit introduction of this discontinuity into the function, however, leads to unnecessary complications in the analytical treatment of the problems before us: it would lead to an infinite value of the derivative $\partial u/\partial x$ at the points of the half plane, which is cumbersome as the actual deviation of the lattice cells from their normal form is finite here and in fact approaches to zero. It is more convenient therefore to consider the function defining u as a function of the coordinates which is continuous also at the half plane $x=0$, $y>0$, and which consequently is continuous through the whole of space, with the exception only of the z-axis; then there is no complication in the expression of deformations by means of the derivatives of this function. It is to be noted, however, that the function thus defined ceases to be a single-valued function: when we describe a closed circuit around the z-axis, considering u as a continuous function of the coordinates, then on coming back to our starting point we shall find that u will have either increased, or decreased by the amount λ_0.

The functions giving the values of v and w, on the other hand, although they likewise cease to have a meaning at the points of the z-axis, are single-valued throughout the whole of space.

The result arrived at can be expressed by saying that the function $u\,(x, y, z)$ possesses a cyclic constant for every closed line embracing the z-axis, which constant has the value λ_0.

The z-axis consequently is to be considered as a singular line for the dislocation, which transforms the space around it into a multiply-connected region [4]).

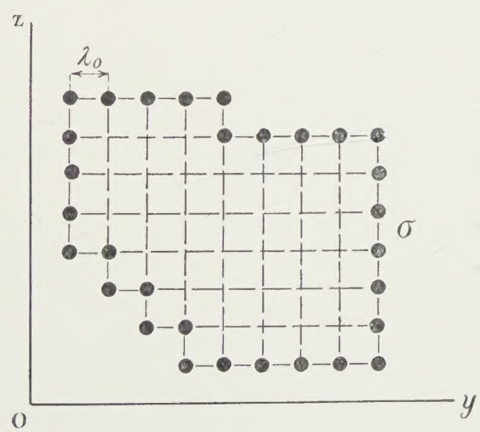

Fig. 2. Extra layer of atoms bounded by an arbitrary line σ in the y, z-plane.

3. Instead of making a cut of half-infinite extent in order to insert into it an extra half plane of atoms, we may also imagine that a cut

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[4]) For a further elucidation of these geometrical relationships the reader is referred to: A. E. H. LOVE, Treatise on the Mathematical Theory of Elasticity (Cambridge 1920), p. 218 seq., and to the paper by VOLTERRA, mentioned in footnote 8) below.

is made over a finite area Σ of the plane $x=0$ (or of a plane $x=$constant), lying wholly inside the crystal, and insert into it an extra layer of atoms. The boundary line σ of the cut, or rather of the extra layer of atoms introduced into the lattice, will consist of segments of rows of atoms, alternately parallel to the y-axis and to the z-axis (compare fig. 2); in the geometrically simplest case it may be a rectangle, but when observed on a scale large compared with the atomic distance λ_0, it may be of any form.

In this case again the components v and w can be represented by single-valued functions of the coordinates, whereas u can be described by a many-valued function, with the cyclic constant λ_0 for every line embracing the boundary line σ, which now is the singular line of the field.

4. It is possible to imagine a dislocation of another character, in which the many-valued function again represents the u-component of the displacement, but in which the singular line is the x-axis. In order to obtain such a case a discontinuity is introduced in the junction of the half-planes $x=const.$, $y<0$ with the half-planes $x=const.$, $y>0$, by making a shift of one atomic distance in passing from the region $z<0$ into the region $z>0$. Then, as indicated schematically in fig. 3, it will be found in moving along a line $x=const.$, $z=const.$, that the component u increases by the amount $\frac{1}{2}\lambda_0$ if $z>0$, whereas it decreases by the same amount if $z<0$.

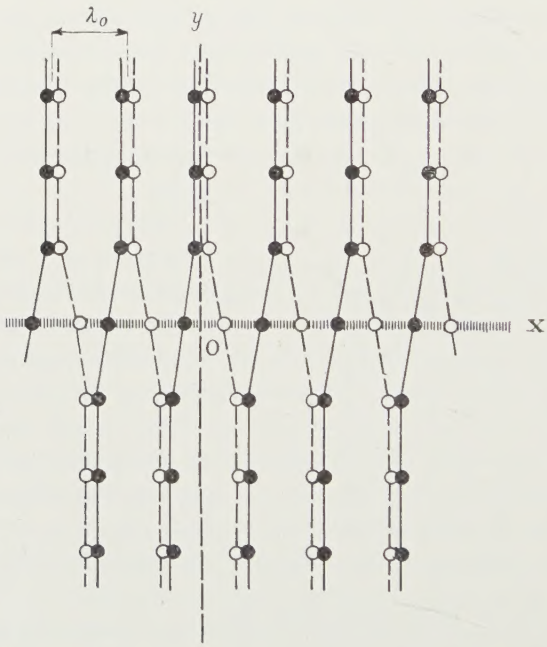

Fig. 3. Schematical picture of a dislocation having the x-axis as singular line. Continuous lines indicate rows of atoms above the x, y-plane; broken lines indicate rows of atoms below this plane.

It must be remarked that the vertical rows of atoms in general will not remain perfectly straight and parallel to the z-axis. From reasons of sym-

metry with respect to the x-axis we might expect that we should observe an increase of u by something like $\frac{1}{4}\lambda_0$ when we move in the direction of $+z$ along a row for which $y<0$; then in moving along a horizontal row in the direction of $+y$ there should be observed a further increase of u by $\frac{1}{4}\lambda_0$; next going downwards along a vertical row for which $y>0$ into the region $z<0$ there should again be an increase by a similar amount, etc. The exact calculation shows that u increases proportionally to the angle described around the x-axis, as will be seen from eq. (27) below.

5. In this way we see that it is possible for singular lines to run parallel to any one of the three coordinate axes. The case last considered may be combined with the other cases; an example is indicated schematically in fig. 4. Here on the right hand side of the plane $x=0$ we have the same

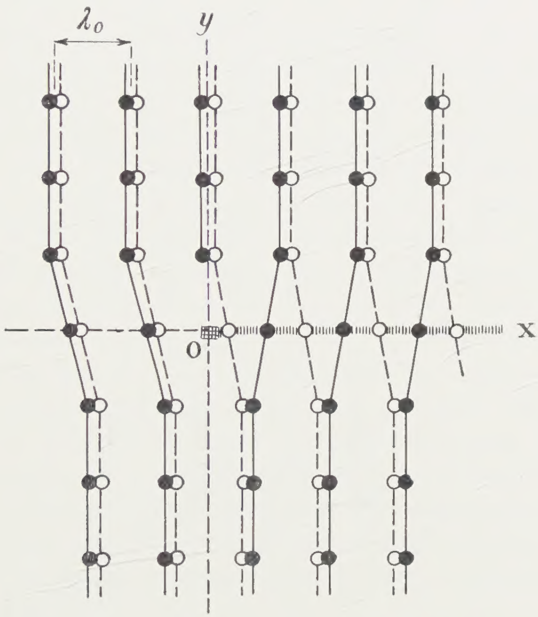

Fig. 4. Schematical picture of a dislocation with a singular line consisting of the positive x-axis together with the positive z-axis (directed upwards perpendicularly to the plane of the paper).

type of dislocation as sketched in fig. 3, with the positive x-axis as the singular line; an extra layer of atoms, however, has been introduced along the quarter plane $x=0$, $y>0$, $z>0$, in consequence of which there are no discontinuities in the region $x<0$, only deformations which will gradually decrease as we go further away in the direction of $-x$. The positive half of the z-axis now has become a singular line, being in fact the continuation of the segment which was formed by the positive x-axis.

Another case is indicated schematically in fig. 5a, which is obtained in the following way: In the x, y-plane a rectangle is imagined with sides $2a$, $2b$ respectively. This rectangle will intersect a number of layers of atoms

which in the undisturbed state of the lattice were parallel to the plane $x = 0$. These layers are assumed to be cut along the lines of intersection (all cuts lying in the plane $z = 0$, and extending from $y = -b$ to $y = +b$); in joining them together again a shift of amount λ_0 has been introduced in the way as indicated in fig. 5*b* (representing a section according to the *x*, *z*-plane). The half plane $x = -a$, $z < 0$ then will possess a free upper border σ_1 along the segment extending from $y = -b$

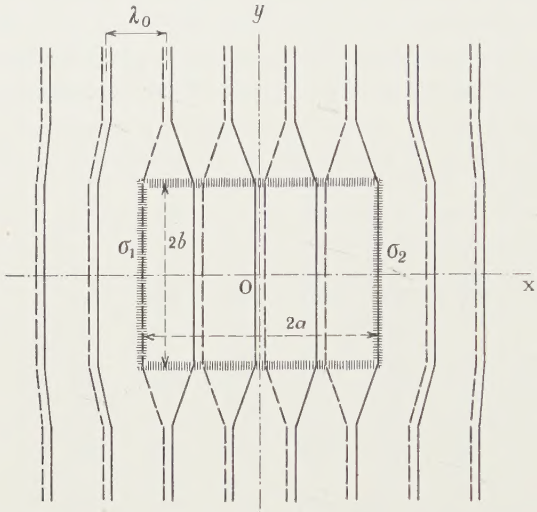

Fig. 5*a*.

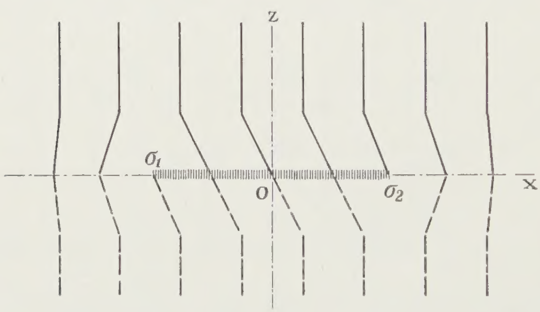

Fig. 5*b*.

Fig. 5. Schematical picture of a dislocation with a singular line in the form of a rectangle in the *x*, *y*-plane. Fig. 5*a*: view in the direction of the negative *z*-axis; fig. 5*b*: section by the plane *Oxz*.

to $y = +b$; the half plane $x = +a$, $z > 0$ has a free lower border σ_2 along a segment of similar extent.

The singular line in this case is formed by the four sides of the rectangle, the segments σ_1 and σ_2 being two of these sides.

It must be stated, of course, that in actual cases the discontinuities possibly may not have the rather simple form assumed in the diagrams given here: there may be regions of irregular atomic arrangement, affecting several rows of atoms in the neighbourhood of what we have called the singular line. However, what is most important in every case is the mode

of connection between the planes or rows of atoms at larger distances from the disturbed region, and for the sake of simplicity in the mathematical formulation it is convenient to keep to the picture of singular lines as determining the geometry of the field.

We now may generalise to cases where the singular line consists of an arbitrary sequence of segments, each of which is parallel to one of the coordinate axes. Again viewing from a scale which is large compared with atomic distances, we may consider such a singular line as being of arbitrary form in space.

One important property of these singular lines, however, must be noticed at once: they can never end at an interior point of the lattice, and must be either closed in themselves, or extend from a point of the exterior surface to another point of this surface or to infinity, or from infinity to infinity.

6. *The field of stress accompanying a dislocation.*—It has been observed by TAYLOR [5]) that although it is not possible to calculate in a rigorous way the forces experienced by the atoms in the immediate neighbourhood of the singular line, at greater distances the mean stresses per unit area can be found with the aid of the equations of the theory of elasticity. In order to arrive at exact results it is necessary to make use of the equations valid for crystalline substances. Even in the case of substances of the regular class these equations are more complicated than those valid for isotropic bodies, the number of constants occurring in them being three, instead of two, while a still greater number occurs in the equations for crystalline substances of other classes [6]). The application of these equations consequently will lead to elaborate expressions, which are not easily handled. It will be useful, therefore, first to develop a provisional treatment, based upon the ordinary theory of elasticity for isotropic bodies, for which the mathematical technique has been built out much further. The results obtained in this way can give an insight into the principal features of the subject [7]), while the application of the exact equations for regular crystals will be considered afterwards in Part II.

The concept of dislocations (originally called "distortions"; the name dislocations is due to LOVE) was introduced into the theory of elasticity by VOLTERRA in order to describe the deformation that can be found in a body occupying a multiply-connected region, when the displacements of the points are given by many-valued functions of the coordinates, there being no exterior forces (neither volume-forces, nor surface-forces) acting on the body [8]). In our case this multiply-connected region is the space

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5) G. I. TAYLOR, *l.c.* p. 375.

6) See A. E. H. LOVE, *l.c.* Chapter VI, and works on physical crystallography.

7) See G. I. TAYLOR, *l.c.* p. 377.

8) V. VOLTERRA, Sur l'équilibre des corps élastiques multiplement connexes, Ann. Ecole Norm. Supér. (3) **24**, p. 400, 1907; A. E. H. LOVE, *l.c.* p. 218. — A report on various types of structural stresses in elastic systems has been given by P. NEMÉNYI

which is obtained when thin cores are cut away from the body along the singular lines considered before.

In the following discussion we assume the crystal to be of infinite extent in all directions, and — unless especially stated — we restrict to singular lines of finite extent, and consequently closed in themselves.

The boundary conditions for the field of stress in this case require that all stress components shall vanish at infinity, in such a way that no resultant force nor resultant moment is transmitted through any plane which recedes to infinity. In virtue of the equations of elastic equilibrium these conditions at the same time ensure that there will be no resultant force or moment acting upon the matter in the immediate neighbourhood of the system of singular lines considered as a whole. It will be natural, however, to introduce the more stringent condition that the resultant force and moment must vanish for any one of the singular lines separately. And generally we must go still further: when the disturbed region along the singular line is of the nature of a relatively thin core, it is inconceivable that forces of considerable magnitude can be transmitted along it from one part to another. Hence, when we consider a small element of this core, bounded *e.g.* by a cylindrical surface having an element of the singular line as axis, the stresses acting on the cylindrical surface must balance each other already very nearly, so that there will remain only residues of an order of magnitude vanishing at the same time as the radius of the cylinder.

7. *General mathematical expressions for the components of the displacement in an elastic body, connected with a given dislocation.* — The general expressions for the displacement components have been deduced by VOLTERRA. However, before giving VOLTERRA's equations we will follow a more synthetic way, which will make clear the meaning of the various terms of these equations.

We start from a set of formulae giving the components of the displacement due to a force operating at a point in an indefinitely extended body [9]). The displacement may be considered as being the sum of two parts, one part having the same direction as the force and being equal in magnitude to the force divided by $4\pi\mu r$, while the second part can be written as the gradient of a certain function ψ: [10])

$$u_k = \frac{F_k}{4\pi\mu r} + \frac{\partial\psi}{\partial x_k} \quad \ldots \ldots \ldots \quad (1)$$

(Selbstspannungen elastischer Gebilde, Zeitschr. f. angew. Math. u. Mech. **11**, pp. 59—70, 1931).

[9]) A. E. H. LOVE, *l.c.* p. 183, eqs. (11). As stated by LOVE at p. 181, these equations originally are due to W. THOMSON.

[10]) For convenience we write x_1, x_2, x_3 for the coordinates; u_1, u_2, u_3 for the components of the displacement, etc. The force is acting at the point ξ_1, ξ_2, ξ_3 and

$$r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2.$$

The quantity μ is one of the two elastic constants (μ, λ) characteristic of an isotropic

the F_k representing the components of the force. For the present it is not necessary to give the "complementary function" ψ explicitly; it must be introduced in order to ensure that the equations of the theory of elasticity shall be fulfilled, but, as we shall see later, this can be done afterwards, so that we are entitled to leave it aside until further consideration. It may be remarked that whereas the first part (the "principal" part) of u_k satisfies the equation

$$\triangle\left(\frac{F_k}{4\pi\mu r}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the "complementary function" ψ is subjected to the equation:

$$\triangle\triangle\psi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now it will have been seen from sections **2—5** that the condition expressing the multi-valuedness of the displacement component u in the cases considered is of the same kind as that of the potential φ associated with the velocity field determined by a *vortex line*, coinciding with the singular line σ. In the hydrodynamical case the cyclic constant of the potential function for every closed line embracing the vortex line is equal to the strength of the vortex line, which thus in our case should be numerically equal to λ_0. — In a more general case, where all three components u_1, u_2, u_3 may be multi-valued, we shall introduce three cyclic constants $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$.

We may, therefore, begin by tentatively writing down the following formula for the "principal" part of the components u_k:

$$u_k^* = \mathfrak{f}_k\,\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where φ is the hydrodynamic potential for a vortex line of unit strength, coinciding with the singular line σ characteristic of the dislocation. The value of φ is equal to the solid angle which a surface Σ bounded by the line σ subtends at the point of the field considered, divided by 4π; it can also be represented by the integral [11]):

$$\varphi=\frac{1}{4\pi}\iint d\Sigma\,\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

ν being the normal to the element $d\Sigma$, drawn in the direction determined by that side of the surface Σ which is considered as the positive side.

8. Formula (5) induces us to interprete the components u_k^* considered in (4) as being due to a system of imaginary *doublets*, distributed over the surface Σ, the axis of the doublets everywhere being normal to $d\Sigma$, whereas the strength (the "moment") of the doublets has the components $\mu\,\mathfrak{f}_k$.

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medium, as used by LOVE and other writers; μ is equal to the shear modulus G, while the ordinary modulus of elasticity (YOUNG's modulus) is given by $E=\mu\,(3\lambda+2\mu)/(\lambda+\mu)$, the compression modulus being $H=\lambda+2\mu/3$. POISSON's ratio $1/m$ of the lateral contraction to the longitudinal extension in an ordinary extension experiment is determined by: $m=2(\lambda+\mu)/\mu$.

[11]) See H. LAMB, Hydrodynamics (Cambridge, 1932), p. 212.

As every doublet consists of two forces of equal and opposite magnitude, the resultant force due to the system is zero. There will be, however, a resultant moment, and it is not difficult to prove that the components of this moment are given by the expressions:

$$\left.\begin{array}{l}\mu\,(A_2 f_3 - A_3 f_2)\\ \mu\,(A_3 f_1 - A_1 f_3)\\ \mu\,(A_1 f_2 - A_2 f_1)\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A_1, A_2, A_3 resp. represent the area's of the projections of Σ upon the three coordinate planes, taken with such signs that $A_k > 0$ when the normal ν to Σ is in the direction of the positive x_k-axis. Our force system consequently does not represent an equilibrium system.

In order to balance this moment, we introduce a system of imaginary forces $\mu\, g_k$ acting at the points of the boundary line, where:

$$\left.\begin{array}{l} g_1 = f_2 \dfrac{d\xi_3}{d\sigma} - f_3 \dfrac{d\xi_2}{d\sigma}\\[2ex] g_2 = f_3 \dfrac{d\xi_1}{d\sigma} - f_1 \dfrac{d\xi_3}{d\sigma}\\[2ex] g_3 = f_1 \dfrac{d\xi_2}{d\sigma} - f_2 \dfrac{d\xi_1}{d\sigma}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is easily proved that this system yields a resultant force equal to zero, while it has a resultant moment which is the exact opposite of that given by eqs. (6). Consequently as a second contribution to the "principal" part of the components u_k we take the expressions:

$$u_k^{**} = \frac{1}{4\pi} \int d\sigma \, \frac{g_k}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The whole system then will be balanced.

It is of importance to observe that formula (8) also can be written in the form of an integral over the surface Σ, as follows [12]:

$$u_k^{**} = \frac{1}{4\pi} \iint d\Sigma \left\{ (\nu k)\, f_l \frac{\partial}{\partial \xi_l}\left(\frac{1}{r}\right) - f_\nu \frac{\partial}{\partial \xi_k}\left(\frac{1}{r}\right) \right\} \quad . \quad . \quad . \quad (9)$$

9. We now turn to the determination of the "complementary function", to be denoted by Ψ. We put:

$$u_k = u_k^{*} + u_k^{**} + \frac{\partial \Psi}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

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[12]) In this equation and in the following ones it is assumed that when in a product or in a differential quotient an index, like l, occurs twice, summation is to be performed with respect to $l = 1, 2, 3$. — The quantities (νk) are the cosines of the angles between the normal ν to $d\Sigma$ and the coordinate axes, and $f_\nu = (\nu l) \cdot f_l$ (component of f_k normal to $d\Sigma$).

The dilatation θ then is given by:

$$\theta = \frac{\partial u_k}{\partial x_k} = \frac{\partial u_k^*}{\partial x_k} + \frac{\partial u_k^{**}}{\partial x_k} + \triangle \Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The components u_k must satisfy the equations of the theory of elasticity:

$$\mu \triangle u_k + (\lambda + \mu) \frac{\partial \theta}{\partial x_k} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As both $\triangle u_k^* = 0$ and $\triangle u_k^{**} = 0$, this equation will be satisfied, provided:

$$\mu \triangle \Psi + (\lambda + \mu)\, \theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

from which it follows that Ψ must satisfy the equation:

$$\triangle \Psi = - \frac{\lambda + \mu}{\lambda + 2\mu} \left(\frac{\partial u_k^*}{\partial x_k} + \frac{\partial u_k^{**}}{\partial x_k} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now from (4), combined with (5), and from (9) it is found that:

$$\frac{\partial u_k^*}{\partial x_k} = \frac{\partial u_k^{**}}{\partial x_k} = - \frac{1}{4\pi} \iint d\Sigma \, \frac{\partial}{\partial \nu} \left\{ \frac{f_l (x - \xi_l)}{r^3} \right\} \quad . \quad . \quad . \quad (15)$$

The solution of (14) therefore can be given in the form:

$$\Psi = - \frac{\lambda + \mu}{4\pi (\lambda + 2\mu)} \iint d\Sigma \, f \, \frac{\partial^2 r}{\partial \nu \, \partial x_l} + \Psi' \quad . \quad . \quad . \quad . \quad (16)$$

where Ψ' is a function which satisfies the equation $\triangle \Psi' = 0$. This function must be determined in such a way that the function Ψ shall not present a discontinuity at the surface Σ. It is found that this is obtained by taking:

$$\Psi' = - \frac{\lambda + \mu}{4\pi (\lambda + 2\mu)} \iint d\Sigma \, \frac{2 f_\nu}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that after a slight reduction there results:

$$\Psi = - \frac{\lambda + \mu}{4\pi (\lambda + 2\mu)} \iint d\Sigma \left[\frac{f_l (\nu\, k)(x_k - \xi_k)(x_l - \xi_l)}{r^3} + \frac{f_\nu}{r} \right] . \quad (18)$$

It is interesting to remark that Ψ also can be represented by a line integral taken along σ. We introduce a vector $\mu\, h_k$ with the components:

$$\left.\begin{aligned} h_1 &= f_2 \frac{x_3 - \xi_3}{r} - f_3 \frac{x_2 - \xi_2}{r} \\ h_2 &= f_3 \frac{x_1 - \xi_1}{r} - f_1 \frac{x_3 - \xi_3}{r} \\ h_3 &= f_1 \frac{x_2 - \xi_2}{r} - f_2 \frac{x_1 - \xi_1}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then it is found that:

$$\Psi = \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} \int d\sigma \left(h_1 \frac{d\xi_1}{d\sigma} + h_2 \frac{d\xi_2}{d\sigma} + h_3 \frac{d\xi_3}{d\sigma} \right) \quad . \quad . \quad . \quad (20)$$

The fact that this transformation is possible proves that Ψ is independent of the particular form given to the surface Σ (provided it is bounded by the line σ), and that consequently Ψ and its derivatives will be continuous at the points of Σ.

Our final expression for u_k thus becomes:

$$u_k = f_k \varphi + \frac{1}{4\pi} \int d\sigma \frac{g_k}{r} + \frac{\partial \Psi}{\partial x_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

with φ given by (5) and Ψ given either by (18) or by (20). The first integral introduces the desired multi-valued character, while all three terms are independent of the particular form given to the surface Σ and exclusively depend upon the form of the boundary line σ.

10. The formulae deduced by VOLTERRA with the aid of a very elegant method, refer to a somewhat more general case than the one considered here [13]. When we restrict to the type of multi-valuedness considered above, these formulae can be given in the form:

$$u_k = \iint d\Sigma \, X_{kl} f_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the X_{kl} ($l = 1, 2, 3$) represent the components of the stress acting on the element $d\Sigma$ at ξ_1, ξ_2, ξ_3, when a unit force in the direction of the x_k-axis is applied at the point x_1, x_2, x_3. When the calculations are worked out, it is found that [14]:

$$\left.\begin{aligned} X_{kl} = \frac{\delta_{kl}}{4\pi} \frac{\partial}{\partial \nu} \left(\frac{1}{r} \right) + \frac{1}{4\pi} \left\{ (\nu k) \frac{\partial}{\partial \xi_l} \left(\frac{1}{r} \right) - (\nu l) \frac{\partial}{\partial \xi_k} \left(\frac{1}{r} \right) \right\} - & \\ - \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} \left\{ \frac{\partial^3 r}{\partial x_k \, \partial x_l \, \partial \nu} - 2 (\nu l) \frac{\partial}{\partial x_k} \left(\frac{1}{r} \right) \right\} & \end{aligned}\right\} . \quad (23)$$

With these values of X_{kl} the expressions (22) are identical with (21).

11. *Application of equations (21) to some simple examples.* — We turn back for a moment to the cases indicated schematically in fig. 1 and 3, although they refer to fields where the singular line is of infinite extent, and will attempt to apply eqs. (21) to them. In these cases $f_2 = f_3 = 0$, while $f_1 = \lambda_0$.

A. In the case of fig. 1 the singular line is the z-axis. In order to find the value of φ by means of eq. (5) we may take an arbitrary half plane for the surface Σ, provided it has the z-axis as its boundary, as different positions of this plane lead to results differing only by an additive constant.

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13) V. VOLTERRA *l.c.* p. 425, eqs. (I), (II).

14) Here $\delta_{kl} = 1$ or 0, accordingly as $k = l$ or $k \neq l$.

It is necessary, however, to specify the positive direction of the normal ν, as this determines the side from which the solid angle must be viewed, and at the same time determines the direction in which the boundary line must be described in the integrals (8) and (20).

When for Σ we take the half plane $x=0$, $y>0$, and as the positive direction of the normal that of the positive x-axis, then the boundary line must be followed in the direction of $-z$; consequently along this line we shall have:

$$d\xi_1/d\sigma = 0; \qquad d\xi_2/d\sigma = 0; \qquad d\xi_3/d\sigma = d\zeta/d\sigma = -1.$$

We now obtain:

(a)
$$\varphi = \frac{1}{2\pi} \operatorname{arctg} \frac{y}{x} + const. \quad \ldots \ldots \quad (24a)$$

(b)
$$g_1 = 0, \qquad g_2 = +\lambda_0, \qquad g_3 = 0.$$

The integral $u_2^{**} = \frac{1}{4\pi} \int d\sigma \frac{g_2}{r}$ is divergent; the relevant part (*i.e.* the part dependent upon x and y), however, can be written:

$$u_2 = \frac{1}{4\pi} \int d\sigma \frac{g_2}{r} \cong -\frac{\lambda_0}{2\pi} \ln \sqrt{x^2+y^2} + const. \quad \ldots \quad (24b)$$

(c)
$$h_1 = 0, \qquad h_2 = -\lambda_0 \frac{z-\zeta}{r}; \qquad h_3 = \lambda_0 \frac{y}{r}.$$

Equation (20) becomes:

$$\Psi = \frac{-(\lambda+\mu)\lambda_0}{4\pi(\lambda+2\mu)} \int d\sigma \frac{y}{r},$$

which, in the same sense as above, gives the result:

$$\Psi \cong \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)}\, y \ln \sqrt{x^2+y^2} + const. \quad \ldots \quad (24c)$$

Hence we obtain [15]):

$$\left.\begin{aligned} u_1 &= \frac{\lambda_0}{2\pi} \operatorname{arctg} \frac{y}{x} + \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)} \frac{xy}{x^2+y^2} \\ u_2 &= -\frac{\mu\lambda_0}{2\pi(\lambda+2\mu)} \ln \sqrt{x^2+y^2} + \frac{(\lambda+\mu)\lambda_0}{2\pi(\lambda+2\mu)} \frac{y^2}{x^2+y^2} \\ u_3 &= 0 \end{aligned}\right\} \quad \ldots \quad (25)$$

B. In the case of fig. 3 the singular line is the x-axis. Along this axis we have $d\xi_2/d\sigma=0$, $d\xi_3/d\sigma=0$; hence the quantities g_k vanish; likewise

15) It may be remarked that these expressions differ from those given by VOLTERRA,

the integral (20) vanishes. There remains only the function φ, which now is equal to:

$$\varphi = -\frac{1}{2\pi} \operatorname{arctg} \frac{z}{y} + const. \qquad (26)$$

Hence [16]):

$$\left.\begin{aligned} u_1 &= -\frac{\lambda_0}{2\pi} \operatorname{arctg} \frac{z}{y} \\ u_2 &= 0 \\ u_3 &= 0 \end{aligned}\right\} \qquad (27)$$

12. *Expressions for the stresses.* — Now that the expressions for the displacement components u_k have been found, the components σ_{kl} of the elastic stresses can be calculated by means of the equations:

$$\sigma_{kl} = \mu \left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) + \delta_{kl} \lambda \theta. \qquad (28)$$

The quantities occurring in these equations can be obtained by means of line-integrals along σ: the terms depending upon the quantities g_k by means of eq. (8); the quantities depending upon Ψ by means of (20); from (11) and (14) it follows that:

$$\theta = -\frac{\mu}{\lambda + \mu} \triangle \Psi \qquad (29)$$

which leads to the equation:

$$\theta = + \frac{\mu}{2\pi(\lambda + 2\mu)} \int d\sigma \left(\frac{h_1}{r^2} \frac{d\xi_1}{d\sigma} + \frac{h_2}{r^2} \frac{d\xi_2}{d\sigma} + \frac{h_3}{r^2} \frac{d\xi_3}{d\sigma} \right) \qquad (30)$$

finally, the derivatives of the potential φ, which in the corresponding hydrodynamic problem represent the components of the velocity, can be

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by way of example, *l.c.* p. 428. When we take $l = \lambda_0$, $m = n = p = q = r = 0$ in those formulae we obtain:

$$\left.\begin{aligned} u_1 &= \frac{\lambda_0}{2\pi} \operatorname{arctg} \frac{y}{x} \\ u_2 &= \frac{\lambda_0}{2\pi} \ln \sqrt{x^2 + y^2} \\ u_3 &= 0. \end{aligned}\right\}.$$

However, the expressions (25) given in the text above are in substantial agreement with the result given by VOLTERRA *l.c.* p. 465, eqs. (I), when in the latter we take $R_1 = 0$, $R_2 = \infty$, and interchange x and y. — The formulae for the stresses given by TAYLOR, Proc. Roy. Soc. London **A 145**, p. 376, 1934, correspond to the expressions given by VOLTERRA at p. 428.

16) This result is in accordance with the formulae given by VOLTERRA at p. 428, if we take $n = -\lambda_0$, $l = m = p = q = r = 0$.

calculated by means of the so-called "formula of BIOT and SAVART" [17]. This is especially convenient when the singular line σ consists of straight segments.

It can be shown that when the line σ is closed in itself and embraces a finite area, the stresses σ_{kl} with increasing distances r become of the order of magnitude at most of r^{-3}.

13. It may be useful to come back to the question of the balancing of the force system alluded to at the end of section **6** and also in section **8**. For this purpose we should consider the resultant force acting upon a cylindrical surface described with the (small) radius a around an element $d\sigma$ of the singular line as axis. However, instead of treating the general problem, we will consider the case of an infinitely extended straight singular line, tangent to the given line at the element $d\sigma$, and having the same cyclic constants as the latter, assuming that when the radius of curvature of the given singular line is great compared with the length of the cylindrical surface to be considered, the stresses for the two cases will differ at most by quantities of a finite order of magnitude, so that the difference between the resultant forces will vanish simultaneously with the radius a.

We first consider the stresses connected with the quantities u_k^* and that part Ψ^* of the complementary function Ψ which can be directly associated with them. In virtue of the relation $\partial u_k^* / \partial x_k = \partial u_k^{**} / \partial x_k$ — see eq. (15) — we have:

$$\Psi^* = \tfrac{1}{2}\,\Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Taking the singular line along the z-axis (the direction being that of $-z$), we again find:

$$\varphi = \frac{1}{2\pi}\, arctg\, \frac{y}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

while, if f_1, f_2, f_3 all are different from zero, we shall have:

$$\left.\begin{aligned} \Psi^* = \tfrac{1}{2}\,\Psi = \frac{-(\lambda+\mu)}{8\pi(\lambda+2\mu)} \int d\sigma \left(f_1 \frac{y}{r} - f_2 \frac{x}{r}\right) \simeq \\ \simeq \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} (f_1 y - f_2 x)\, ln \sqrt{x^2+y^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

From this expression for Ψ^* we obtain:

$$\theta = -\frac{\mu}{\lambda+\mu} \triangle \Psi^* = -\frac{\mu}{2\pi(\lambda+2\mu)} \frac{f_1 y - f_2 x}{x^2+y^2} \quad . \quad . \quad . \quad (34)$$

Writing:

$$r = \sqrt{x^2+y^2}; \qquad\qquad a = arctg\; y/x \quad . \quad . \quad . \quad (35)$$

it is found that the components σ_{rl} of the stresses acting upon an element of the cylindrical surface can be calculated from the equation:

$$\left.\begin{aligned} \sigma_{rl} &= \sigma_{1l} \cos a + \sigma_{2l} \sin a = \\ &= \mu (f_1 \cos a + f_2 \sin a) \frac{\partial \varphi}{\partial x_l} + 2\mu \frac{\partial^2 \Psi^*}{\partial r\, \partial x_l} + \lambda \theta \cos(rl) \end{aligned}\right\} \quad . \quad . \quad (36)$$

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[17]) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 211, eqs. (2) and (3).

which gives:

$$\left.\begin{aligned}\sigma_{rx} &= -\frac{\mu}{2\pi a}(f_1 \cos a \sin a + f_2 \sin^2 a) - \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)a} f_2 - \\ &\qquad - \frac{\mu\lambda}{2\pi(\lambda+2\mu)a}(f_1 \cos a \sin a - f_2 \cos^2 a), \\ \sigma_{ry} &= +\frac{\mu}{2\pi a}(f_1 \cos^2 a + f_2 \cos a \sin a) + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)a} f_1 - \\ &\qquad - \frac{\mu\lambda}{2\pi(\lambda+2\mu)a}(f_1 \sin^2 a - f_2 \cos a \sin a), \\ \sigma_{rz} &= 0. \end{aligned}\right\} \quad . \quad . \quad (37)$$

The resultant force acting from the elastic medium upon the cylindrical element consequently has the components (per unit length):

$$\left.\begin{aligned} &\text{in the } x\text{-direction: } -\mu f_2 \\ &\text{in the } y\text{-direction: } +\mu f_1 \\ &\text{in the } z\text{-direction: } \quad 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

14. We must next consider the stresses connected with the quantities u_k^{**} and the second part Ψ^{**} of the complementary function. We have:

$$\left.\begin{aligned} \mu g_1 &= -\mu f_2 \\ \mu g_2 &= +\mu f_1 \\ \mu g_3 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

while, from (8):

$$u_k^{**} = \frac{g_k}{4\pi}\int \frac{d\sigma}{r} \cong -\frac{g_k}{2\pi} \ln r \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

As $\Psi^{**} = \Psi^*$, $\theta^{**} = \theta^*$, the calculation is not difficult; instead of (36) we obtain:

$$\sigma_{rl} = -\frac{\mu g_l}{2\pi r} - \frac{\mu(g_1 \cos a + g_2 \sin a)}{2\pi}\frac{\partial}{\partial x_l}(\ln r) + 2\mu \frac{\partial^2 \Psi^*}{\partial r\, \partial x_l} + \lambda\theta \cos(rl) \quad . \quad (41)$$

and the components of the resultant force become:

$$\left.\begin{aligned} &\text{in the } x\text{-direction: } +\mu f_2 \\ &\text{in the } y\text{-direction: } -\mu f_1 \\ &\text{in the } z\text{-direction: } \quad 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

These quantities are equal and opposite to those given in (38); hence there is no resultant force acting upon our cylindrical element, as had been required in section 6.

It may be remarked that the stress component σ_{zz} in general will not be zero, so that there may be tensions and pressures in the direction of the singular line, connected with the dilatation θ. The mean value of σ_{zz} over a cross section of the element, however, vanishes.

15. *Further geometrical considerations. Migration of a dislocation through the atomic lattice.* — The passage of a two-dimensional unit

dislocation across a crystal has been described by TAYLOR as the basic phenomenon in the explanation of the process of slipping [18]. A schematical picture of a case in which the singular line σ is parallel to the y-axis has been given in fig. 6; it will be seen that in the situation of fig. 6*a* the atom 1 can jump over to another equilibrium position; next the atom 2 can

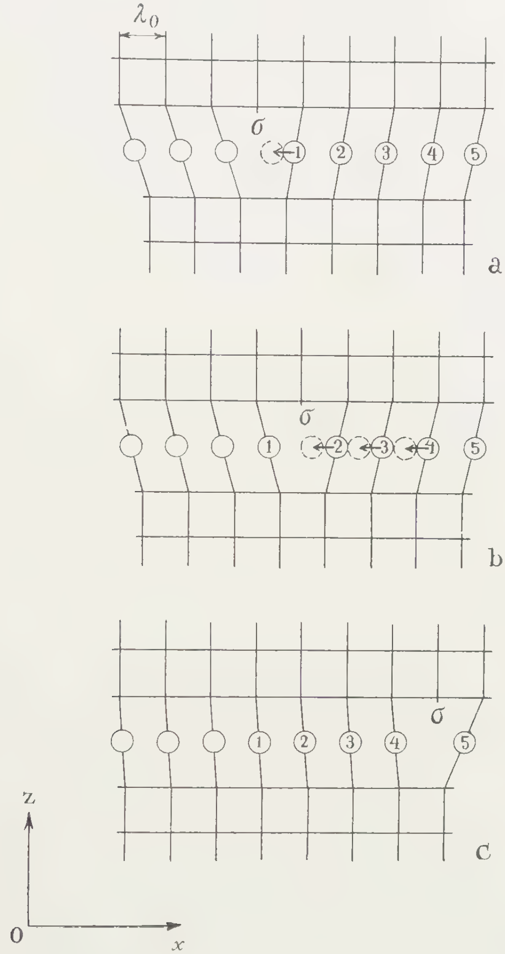

Fig. 6. Schematical picture of the migration of an elementary dislocation through the lattice.

make a similar jump, then 3, 4, ..., and in consequence of these jumps the singular line σ moves to the right. In a crystal of finite dimensions this process will be accompanied by a shift in the relative position of the upper and the lower parts of the crystal; this shift is of such a magnitude that it becomes equal to the amount λ_0 when the dislocation has moved across the whole crystal from the left hand boundary to the right hand boundary.

[18]) G. I. TAYLOR, *l.c.* p. 368. — See also "First Report on Viscosity and Plasticity", p. 199.

When the dislocation has migrated only over a distance L, in a crystal the dimension of which in the same direction is a, then the relative shift of the two parts will be given by $\lambda_0 L/a$.

When the dislocations are characterized by closed singular lines of finite extent, embedded in a lattice of indefinite, or at any rate of very large extent, it is to be noted that at large distances from the dislocation the lattice must be wholly regular. When we imagine a closed surface surrounding the characteristic lines of all dislocations, outside of this surface the u_k will be single-valued functions, which with increasing distances either will become zero or will approach to constant values. It will be evident that in a region with regular structure no dislocation can be generated "out of nothing": dislocations either must have been originated during the process of growth of the crystal, or they must have been derived from other already existing regions of irregular structure. When a dislocation of the type considered by us takes its birth from some unspecified region of irregular structure, then — the same as in any other case — the condition of never having an open end in the interior of the lattice always will remain valid: the singular line characteristic of the dislocation either must be closed in itself, or else its ends must be situated at the boundary of the irregular region (or eventually perhaps in the interior of this region). It would appear probable that a given dislocation can be displaced through the lattice over an arbitrary distance, possibly in various directions. The character of being a closed line will not be lost during such a displacement, although the singular line perhaps may change of form; further the strength of the dislocation, or more exactly the values of the cyclic constants $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ associated with the singular line, will not change. When a singular line in its migration through the lattice should meet another singular line, then it is to be expected that the simple type of migration, determined by jumps of the atoms of the kind as described above, cannot be continued. Hence we may assume that two singular lines in their process of migration in general cannot cross each other, or at least will have a certain difficulty in crossing each other. (It may be that the approach of the two dislocations leads to the formation of a region of irregular structure of larger extent, from which, under suitable circumstances, a new dislocation may take its birth; a more detailed investigation of such a process will be useful, but probably may be difficult). At any rate we may suppose that the easy migration of a dislocation is impeded when it meets other dislocations; this is one of the features which serve as a basis for the explanation of the fact that the plastic deformation of a crystal gradually becomes more difficult (*i.e.* requires the application of greater forces) with increasing values of the shear.

It will be evident that in the points mentioned there is an analogy with the properties of vortices in an ideal liquid. We might even go further and ask whether *e.g.* processes in which there is a change in the area enclosed by the singular line (or more exactly in the area's A_1, A_2, A_3 of its three

projections upon the coordinate planes; compare section **8**) will require the application of exterior forces to the lattice. We shall come back to this point in section **19**.

However, although a certain analogy with vortex lines exists, we must not forget that the migration of a dislocation, as pictured schematically in fig. 6, is intimately connected with the geometry of the atomic lattice. Consequently there may be restrictions on the possibilities for the displacement of the singular line, which have no analogy in the hydrodynamic case.

16. There are a few examples of migrations which can be discussed in a simple way.

Consider the case pictured in fig. 7. In the first place this may be

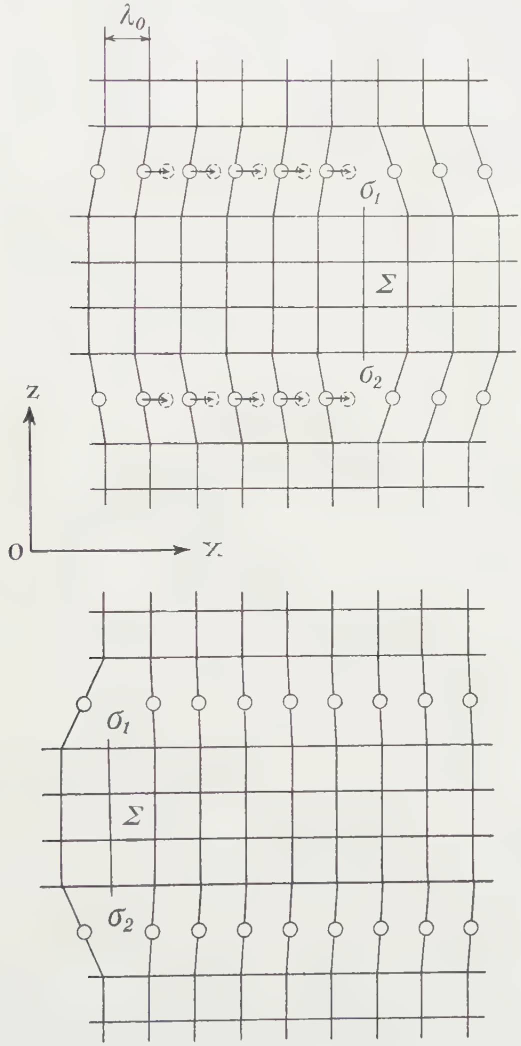

Fig. 7. Schematical picture of the migration of a dislocation, bounded by two parallel singular lines, when these lines move simultaneously with equal steps.

considered as representing a section of a lattice in which a dislocation has been introduced by inserting into it an extra layer of atoms along a surface Σ, perpendicular to the x-axis and bounded by two parallel singular lines σ_1 and σ_2 perpendicular to the plane of the paper (*i.e.* parallel to the y-axis), both of infinite extent. As indicated schematically, it is possible in such a case that the two singular lines migrate simultaneously over the same distance, *e.g.* to the left. It will be seen that, although there is a relative shift of the central portion of the lattice (*i.e.* the portion situated between the planes described by the singular lines in their movement) with respect to the rest, there is no resulting shift of the uppermost portion with respect to the undermost.

Fig. 7 may be considered also as representing a section of a three-dimensional field, in which the surface Σ is of finite extent and is bounded by a closed line σ (Σ in this case still being plane and perpendicular to the x-axis). There seems to be no objection to the assumption that in such a case a similar migration is possible, jumps of atoms now taking place with equal frequency at all points of σ; the singular line then migrates without change of form through the lattice in the direction perpendicular to its plane.

Turning back to the original conception in which we had two parallel singular lines σ_1 and σ_2 of infinite extent, it will be evident that the parallel and equal migration of σ_1 and σ_2 is not a necessary feature: these lines may just as well move independently of each other, *e.g.* in the way as indicated in fig. 8. The dislocation then of course obtains a different character in so far as the surface Σ of fig. 7 does no longer exist.

Is it possible to imagine something to be compared with the latter case taking place when the surface Σ is bounded by a closed line σ?

Referring to fig. 9, where the line σ originally had the form of a rectangle $ABDE$ (the plane of the rectangle being perpendicular to the x-axis), we may imagine that jumps of atoms take place only along the parts CD, DE and EF of σ, producing a displacement of $CDEF$ parallel to itself towards a new position $C'D'E'F'$, whereas the part $FABC$ remains where it was. An irregularity in the arrangement of the atoms then will be produced along the whole course of the lines CC' and FF', and these lines in fact will become parts of the singular line characteristic of the dislocation in its new form, joining up the parts $FABC$ and $C'D'E'F'$. The nature of the singularity in the immediate neighbourhood of the segments CC' and FF' is of the character indicated in fig. 3 (and also in fig. 5 for the parts of the singular line which are parallel to the x-axis), while in the neighbourhood of the point F it can be compared to the case indicated in fig. 4.

17. The picture arrived at in the preceding section can be of help in discussing a point which had been raised in some sections devoted to the phenomena of plastic deformation in crystalline substances of the "Second Report on Viscosity and Plasticity". In connection with views brought forward by TAYLOR it had been assumed in the "First Report" that when

in the course of a shearing process applied to a crystal, a certain number of dislocations have started from already existing flaws, and have moved

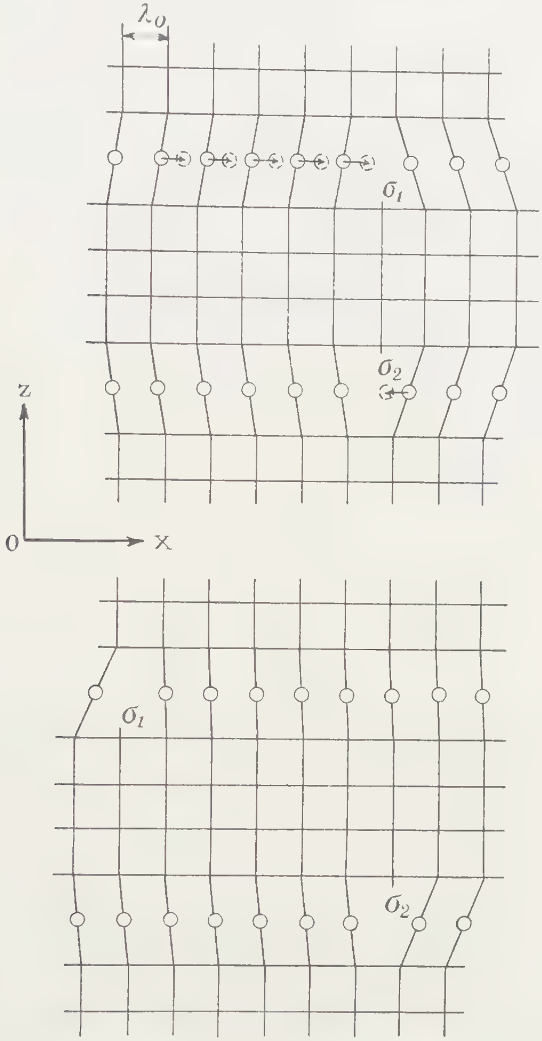

Fig. 8. Schematical picture of the change of character of a dislocation bounded by two parallel singular lines, when these lines move in opposite directions.

through the lattice until they are stopped by encountering other regions of irregular structure, there is produced a field of stresses, formed by the resultant effect of the fields connected with these dislocations, which field counteracts the stress due to the exterior forces causing the shearing process [19]. In making an estimate of the magnitude of the average shearing stress derived from the fields of the dislocations, TAYLOR's picture and calculations were used, referring to the two-dimensional type of dislo-

[19]) "First Report on Viscosity and Plasticity", p. 209.

cations, all singular lines being parallel to each other and extending right across the crystal. The average shearing stress then is found to be inversely proportional to the mean distance between the singular lines; consequently it is directly proportional to the square root of the number of singular lines per unit area.

The picture of a system of parallel dislocation lines, all extending in the lateral direction right across the crystal, however, has a degree of regularity which appears greater than may be expected in a crystal with flaws, and it is therefore that in the "Second Report a "three-dimensional" picture with dislocations of finite lateral extent had been suggested [20]).

We may now attempt to consider the process indicated schematically in fig. 9 as a possible case of the migration of such a three-dimensional dislocation. The case of fig. 7, which as stated, likewise can be taken as

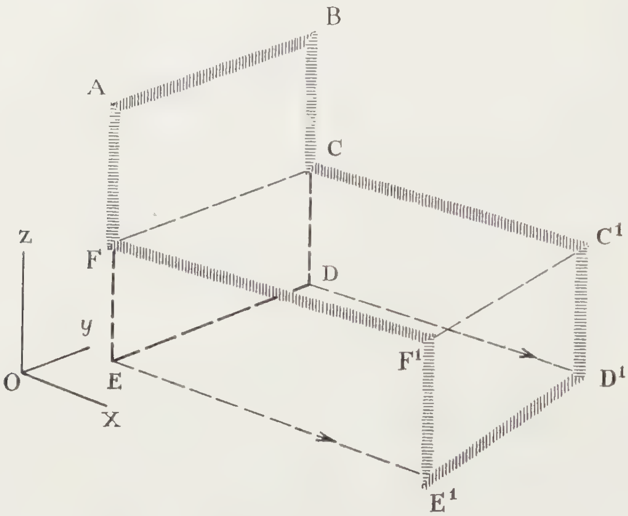

Fig. 9. Schematical picture of the transformation of a dislocation, originally characterized by a rectangular singular line situated in a plane parallel to the y, z-plane.

representing a dislocation of finite extent, is of no use, as in this case no resultant average shear will appear in the crystal. In the case of fig. 9 on the other hand, there is a contribution to the average shear, depending in magnitude upon the area of the rectangle $FF'C'CF$. The lateral extension of the dislocation, determined by the length AB ($= FC = ED$), like the dimension AE or BD, will depend upon the dimensions of the disturbed region from which this dislocation originated, and thus will correspond to the quantity l in the equations of the "Second Report"; whereas the distance CC' or FF' represents the length L of the path described by the dislocation [21]).

[20]) "Second Report on Viscosity and Plasticity", pp. 200 seq.

[21]) Compare "Second Report", p. 202, eq. (5. 1b), where λ corresponds to λ_0 in the present communication.

In calculating the field of stress associated with the type of singular line pictured in fig. 9 we may — in a similar way as can be done in all problems relating to vortex lines — separately consider the three circuits $AFCBA$, $F'E'D'C'F'$ and $FF'C'CF$. When we ask for the magnitude of the stresses in points situated at distances, say r_1 from $AFCBA$ and r_2 from $F'E'D'C'F'$, which are large in comparison with the sides AB and AF or $F'E'$ etc., then according to what has been remarked at the end of section **12**, the contributions of these circuits into the stresses will be of the orders $(r_1)^{-3}$ and $(r_2)^{-3}$ respectively. The contribution of the circuit $FF'C'CF$, however, will be of a different order when the length $CC' = L$ itself is great compared with $F'C' = l$.

18. *Calculation of the field of stress connected with a singular line of a form as given in fig. 9.* — With reference to fig. 10 the calculation for the circuit $FF'C'CF$ can be given as follows: Let φ, as before, represent the

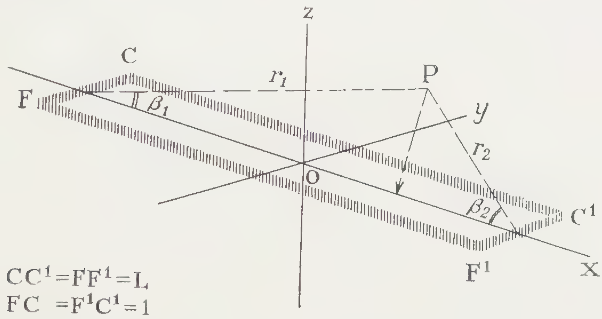

Fig. 10. The rectangular circuit $FF'C'CF$ of fig. 9 considered separately.

solid angle, divided by 4π, which the rectangle $FF'C'CF$ subtends at the point P. The positive direction of the normal to the rectangle is the direction Oz; the same as with the dislocation in its original form, characterized by $AEDBA$, there is a discontinuity in the component u_1 only, so that $f_2 = f_3 = 0$, while $f_1 = \lambda_0$. Hence, according to (4):

$$u_1^* = \lambda_0\, \varphi, \quad u_2^* = u_3^* = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

Along FF' and CC' we have $g_1 = g_2 = g_3 = 0$ by (7), so that these segments do not give a contribution to u^{**}. From (19) and (20) it follows that they neither give a contribution to the value of Ψ.

Along $F'C'$ and CF (the positive direction of integration along the circuit being $FF'C'CF$) we have $g_1 = g_2 = 0$; along $F'C'$ we have: $d\xi_2/d\sigma = +1$; $g_3 = +\lambda_0$; $h_2 = -\lambda_0 z/r_2$, and along FC: $d\xi_2/d\sigma = -1$; $g_3 = -\lambda_0$; $h_2 = -\lambda_0 z/r_1$. Hence we obtain:

$$\text{from (8):} \quad u_1^{**} = u_2^{**} = 0; \qquad u_3^{**} = \frac{\lambda_0\, l}{4\pi}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \quad . \quad . \qquad (44)$$

$$\text{from (20):} \quad \Psi = -\frac{\lambda + \mu}{\lambda + 2\mu}\,\frac{\lambda_0\, l}{4\pi}\left(\frac{z}{r_2} - \frac{z}{r_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

It has been assumed in these expressions that r is large compared with l.

The stress τ which had been considered in the "First" and "Second Reports" corresponds to the component σ_{13} in the notation used here, which is given by:

$$\sigma_{13} = \mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) = \mu\left(\lambda_0 \frac{\partial \varphi}{\partial z} + \frac{\partial u_3^{**}}{\partial x} + 2\,\frac{\partial^2 \Psi}{\partial x\,\partial z}\right) \quad . \quad . \quad (46)$$

In this expression $\partial^2 \Psi/\partial x \partial z$ is to be calculated from (45); $\partial\varphi/\partial z$ is obtained by calculating the velocity field associated with the rectangular vortex line. With sufficient approximation we have:

$$\mu\left(\lambda_0 \frac{\partial \varphi}{\partial z} + \frac{\partial u_3^{**}}{dx}\right) = \frac{\mu\,\lambda_0 l}{4\pi}\,\frac{y^2 - z^2 - l^2/4}{(y^2+z^2+l^2/4)^2 - y^2\,l^2}(\cos\beta_1 + \cos\beta_2) \quad . \quad (47)$$

the angles β_1 and β_2 being defined in fig. 10.

Even without calculating the value of $\partial^2\Psi/\partial x\partial z$ (which for $r \gg l$ becomes of the order r^{-3}, as mentioned before), it will be seen from the expression (47) that so long as $|x|$ does not greatly surpass $L/2$, so that $\cos\beta_1 + \cos\beta_2$ is either > 1, or not much below 1, this result leads to values of σ_{13} of an order of magnitude decreasing inversely proportional to the square of the distance from the x-axis, *i.e.* it leads to values which for constant y/z practically are inversely proportional to $(y^2 + z^2)$. The dimensions of the region in which this result holds of course depend upon the magnitude of L.

19. It is of interest also to consider the stresses σ_{12} and σ_{13} in the points of the plane determined by the rectangle $F'E'D'C'$. We must then work out the calculations for the singular line $FF'E'D'C'C$; when L is sufficiently great, we may neglect the contributions due to the part $CBAF$. It will be superfluous to give the calculations in detail; and we mention only the following points:

It is found again that $u_1^{**} = 0$, whereas:

$$\left.\begin{aligned} u_2^{**} &= \frac{\lambda_0}{4\pi}\left(\int_{E'}^{F'} \frac{d\zeta}{r} - \int_{D'}^{C'} \frac{d\zeta}{r}\right) \\ u_3^{**} &= \frac{\lambda_0}{4\pi}\int_{E'}^{D'} \frac{d\eta}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Further:

$$\Psi = \frac{(\lambda+\mu)\,\lambda_0}{4\pi\,(\lambda+2\,\mu)}\left\{\int_{D'}^{C'} d\zeta\,\frac{y-\eta}{r} - \int_{E'}^{F'} d\zeta\,\frac{y-\eta}{r} - \int_{E'}^{D'} d\eta\,\frac{z-\zeta}{r}\right\} \quad . \quad (49)$$

In the equations for the stress components σ_{12} and σ_{13} the derivatives

$\partial^2 \Psi/\partial x \partial y$ and $\partial^2 \Psi/\partial x \partial z$ occur; as $\partial \Psi/\partial x = 0$ for points in the plane of the rectangle $F'E'D'C'$, the contributions to be derived from Ψ vanish here.

The values of $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ again can be obtained by means of the formulae for the velocity field associated with a vortex line. It is found that the sum

$$\lambda_0 \frac{\partial \varphi}{\partial y} + \frac{\partial u_2^{**}}{\partial x},$$

occurring in the expression for σ_{12}, is determined exclusively by the contributions to the value of $\partial\varphi/\partial y$ derived from the lines FF' and $C'C$; whereas the sum

$$\lambda_0 \frac{\partial \varphi}{\partial z} + \frac{\partial u_3^{**}}{\partial x},$$

occurring in the expression for σ_{13}, in the same way is determined by the contributions to the value of $\partial\varphi/\partial z$ derived from these lines.

The values of the stress components σ_{12} and σ_{13} in the points of the plane determined by $F'E'D'C'$ finally become:

$$\left.\begin{aligned} \sigma_{12} &= \frac{\mu \lambda_0 l}{4\pi} \frac{-2yz}{(y^2+z^2+l^2/4)^2 - y^2 l^2} \\ \sigma_{13} &= \frac{\mu \lambda_0 l}{4\pi} \frac{y^2 - z^2 - l^2/4}{(y^2+z^2+l^2/4)^2 - y^2 l^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Let us give attention to the value of σ_{13} at the points of the segment $E'D'$. It is not difficult to calculate the mean value of σ_{13} at the points of this segment; this mean value is found to be:

$$(\sigma_{13})_m = -\frac{\mu \lambda_0}{4\pi l} \ln \frac{l^2 + h^2}{h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and thus appears to be *negative*.

In order to understand the meaning of the sign of σ_{13}, we remark that in the cases indicated in figs. 6 and 8 the application of a positive exterior shearing stress to the system, acting to the right at the upper surface of the crystal, and to the left at the lower surface, will promote the occurrence of the type of migration pictured in these diagrams. The case of fig. 9 has been derived from that of fig. 8 without change of signs, so that the same result will apply to it. Hence we may conclude that the appearance of a negative value of $(\sigma_{13})_m$ along $E'D'$ will act in the opposite way, and consequently will drive back the migration process, or at any rate impede its further progress in the original direction.

Here thus we have an instance of the "counteracting" effect of the field of stress connected with the dislocation itself. It will be evident that this "counteracting" effect can be overcome by the application to the crystal of an exterior shearing stress τ of sufficient positive magnitude.

We now are in a position to construct an expression for the work that apparently must be performed in order to displace the part $F'E'D'C'$ of the singular line in the positive x-direction, or in other words to increase the length L of the lines FF' and CC'. From eq. (5. 1b), p. 202 of the "Second Report", we see that the increase of the mean shear of a rectangular crystal with sides a, b, c due to an increase of L is given by:

$$d\gamma_1/dL = \lambda_0\, l/a\, b\, c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

The work done in this process by the exterior shearing force τ is determined by the product:

$$\tau \,.\, (d\gamma_1/dL) \,.\, (\text{volume});$$

hence, with $\tau = -(\sigma_{13})_m$, we obtain:

$$\text{work per unit increase of } L = \frac{\mu\,\lambda_0^2}{4\pi}\, ln\, \frac{l^2+h^2}{h^2} \quad . \quad . \quad . \qquad (53)$$

It must be remarked that these considerations are given only in order to illustrate some of the concepts which have arisen in considering the phenomena accompanying the migration of a dislocation through the lattice; no great value can be attached to the exact form of the equations derived. For instance we have left aside the effect which the stress σ_{12} may have at the segments $F'E'$ and $C'D'$; it is true that the values of σ_{12} at these two segments are of opposite sign, and thus do not call for compensation by the application of an exterior shearing stress which can store work in the system. It will be a matter for further speculation to find out what is the meaning of the relations which have turned up here.

20. The impression will have been obtained that the introduction of dislocations of three-dimensional type leads to a picture possessing at least some features which point in the direction of the assumptions of which use was made in the "Second Report" [22]. Consequently we might imagine that in a crystal subjected to shear in the x-direction, along planes parallel to the x, y-plane, there would appear a number of disturbed strips of the nature of the rectangle $FF'C'CF$ in fig. 9 or fig. 10, extending over various lengths L in the x-direction, and having breadths l in the y-direction. A schematical picture of such a system of strips has been presented in fig. 11.

Nevertheless, the picture does not give results ready for immediate use in further calculations. In the preceding section we had arrived at an instance of a counteracting field, impeding the progress of a dislocation. From eqs. (47) and (50) it can be derived that in a region of the lattice surrounding the dislocation the value σ_{13} remains negative so long as $|\,y\,|$ is smaller than $|\,z\,|$. Hence the counteracting effect is also to be observed in the regions situated directly above and below the dislocation

[22]) "Second Report", p. 204 in connection with eq. (5. 12b).

considered. However, in regions situated further away in the lateral direction, where $|y|$ becomes greater than $|z|$, the effect is of opposite sign,

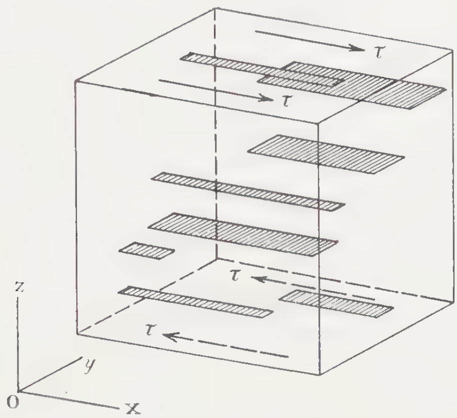

Fig. 11. Crystal block with a number of rectangular dislocation lines parallel to the x, y-plane.

and with the aid of eq. (46) it is not difficult to prove that the mean value of σ_{13} over a plane $z = const.$ is equal to zero.

This result is connected with the circumstance that the mathematical considerations developed above refer to the case of dislocations of finite extent embedded in a lattice which at great distances ultimately approaches to perfect regularity. Clearly such a picture is idealized too far, and cannot represent the state of things in an actual crystal, exhibiting irregularities of growth, etc. We must assume that in an actual crystal many of the irregularities, whatever be their nature, will extend over great distances, and that they will meet each other at various places, so that the crystal is divided up into more or less separate regions of regular structure. The boundary between two adjacent regions of regularity in many cases may be formed by systems of irregularities arranged so as to form a surface, and the dislocations migrating through the crystal in a process of plastic deformation often will have their endpoints moving over such surfaces. At these surfaces moreover the displacement components u_k may be subjected to certain conditions, which will react upon the field of stress.

21. *Systems of dislocations forming a "surface of misfit" between two regions of a lattice.* — An adequate treatment of the problems touched upon in the preceding section is difficult, and will not be given here. We shall restrict to the investigation of a surface of irregularity built up from a system of simple rectilinear dislocations of the type indicated in fig. 1. We assume a two-dimensional field, all singular lines being perpendicular to the x, y-plane. The problem to be considered is closely related to one treated by TAYLOR in order to obtain an example of a "surface of misfit" [23]).

[23]) G. I. TAYLOR, *l.c.* p. 400.

Making use of the equations developed in section **11 A**, we shall write, with a slight change of notation:

$$\left.\begin{aligned} u &= u^* + \partial\Psi/\partial x \\ v &= v^* + \partial\Psi/\partial y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

When the complex variable $x + i\,y$ (with $i = \sqrt{-1}$) is introduced, eqs. (24*a*), (24*b*) give:

$$u^* + i\,v^* = \frac{-i\,\lambda_0}{2\pi}\,ln\,(x + i\,y) + const. \quad . \quad . \quad . \quad . \quad (55)$$

which is a form convenient for generalisation.

We take the case in which there are singular lines at the infinite series of points:

$$x = n\,l, \qquad y = n\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where l and h are two arbitrary constants (both being equal to some multiple of λ_0), while n takes all integer values from $-\infty$ to $+\infty$. By giving special values to l and h various subcases can be constructed; with $h = 0$ all singular lines are situated in a horizontal plane (in the x, z-plane), with $l = 0$ all are situated in a vertical plane (in the y, z-plane). — Every singular line gives a contribution into u^* and v^* which can be expressed by:

$$(u^* + i\,v^*)_n = -\frac{i\,\lambda_0}{2\pi}\,ln\,\frac{x + i\,y - n\,l - i\,n\,h}{n\,(l + i\,h)} \quad . \quad . \quad . \quad . \quad (57)$$

According to well known procedures applied in the theory of functions of a complex variable, we consequently may construct the solution of our problem with its infinite number of singular lines by writing:

$$u^* + i\,v^* = -\frac{i\,\lambda_0}{2\pi}\,ln\,sin\,\frac{\pi\,(x + i\,y)}{l + i\,h} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

For convenience in notation we introduce the auxiliary variables:

$$\frac{\pi\,(l\,x + h\,y)}{l^2 + h^2} = \xi, \qquad \frac{\pi\,(l\,y - h\,x)}{l^2 + h^2} = \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

The separation of real and imaginary parts then can be effected by writing [24]):

$$sin\,(\xi + i\,\eta) = M\,e^{i\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60a)$$

where:

$$M = \sqrt{sin^2\,\xi + sinh^2\,\eta}, \quad tg\,\alpha = \frac{tgh\,\eta}{tg\,\xi} \quad . \quad . \quad . \quad . \quad (60b)$$

so that:

$$u^* = \frac{\lambda_0\,\alpha}{2\pi}; \qquad v^* = -\frac{\lambda_0}{2\pi}\,ln\,M \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

[24]) Compare e.g. E. JAHNKE u. F. EMDE, Funktionentafeln, 1st Ed. (Leipzig u. Berlin 1909), p. 11; 2nd Ed. (1933), p. 60.

From the theory of functions of a complex variable it follows that:

$$\frac{\partial u^*}{\partial x} = \frac{\partial v^*}{\partial y}\ ; \qquad \triangle\, u^* = \triangle\, v^* = 0.$$

Equation (14) thus becomes:

$$\triangle\, \Psi = -\frac{2(\lambda+\mu)}{\lambda+2\mu}\,\frac{\partial u^*}{\partial x} = \frac{\lambda+\mu}{\lambda+2\mu}\;\frac{\lambda_0}{2(l^2+h^2)}\;\frac{h \sin 2\xi + l \sinh 2\eta}{M^2}. \qquad (62)$$

It is difficult to find the expression for Ψ itself, but the problem is satisfied if we take [25]:

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial \xi} &= \frac{\lambda+\mu}{\lambda+2\mu}\;\frac{\lambda_0}{2\pi^2}\left\{\frac{\eta\,(l \sin 2\xi - h \sinh 2\eta)}{2M^2} + h \ln M\right\} \\ \frac{\partial \Psi}{\partial \eta} &= \frac{\lambda+\mu}{\lambda+2\mu}\;\frac{\lambda_0}{2\pi^2}\left\{\frac{\eta\,(l \sinh 2\eta + h \sin 2\xi)}{2M^2} + l \ln M\right\} \end{aligned}\right\} . \qquad (63)$$

From these equations $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ can be obtained without difficulty.

22. It is of interest first to investigate the meaning of these equations for points at large distances from the row of singular lines. As the row itself is situated at the line $\eta = 0$, these points are obtained by considering large values of η. We must distinguish between positive and negative values, and it is useful to observe that η is positive on the left hand side and negative on the right hand side.

For *positive* η we find:

$$M \simeq \sinh \eta\ ;\ \ln M = \text{const.} + \eta$$

$$\operatorname{tg} \alpha = \cot \xi \quad ; \qquad \alpha = \text{const.} - \xi\ ;$$

hence, neglecting constant amounts:

$$\left.\begin{aligned} u^* &= -\frac{\lambda_0\,\xi}{2\pi} = -\frac{\lambda_0\,(lx + hy)}{2(l^2+h^2)} \\ v^* &= -\frac{\lambda_0\,\eta}{2\pi} = -\frac{\lambda_0\,(ly - hx)}{2(l^2+h^2)} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \qquad (64)$$

Further:

$$\frac{\partial \Psi}{\partial \xi} \simeq 0\ ; \qquad \frac{\partial \Psi}{\partial \eta} = \frac{\lambda+\mu}{\lambda+2\mu}\;\frac{\lambda_0\, l\eta}{\pi^2} \quad . \; . \; . \; . \; . \qquad (65)$$

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25) Equation (62) does not wholly determine the function Ψ, as a function Ψ', satisfying the equation $\triangle\, \Psi' = 0$, always can be added. The value of Ψ' can be fixed only by having recourse to the conditions assumed at infinity. In the expressions (63) a form of Ψ has been chosen, which leads to the most satisfactory behaviour of u and v at infinity, as will be seen from the results given in section **22**.

from which:

$$\left.\begin{aligned}\frac{\partial\Psi}{\partial x} &\simeq -\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, l\, h\,(l y - h x)}{(l^2+h^2)^2}\\ \frac{\partial\Psi}{\partial y} &\simeq +\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\lambda_0\, l^2\,(l y - h x)}{(l^2+h^2)^2}\end{aligned}\right\} \quad \ldots\ldots \quad (66)$$

For *negative* η the values of u^*, v^*; $\partial\Psi/\partial\xi$ and $\partial\Psi/\partial\eta$; $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ change sign.

In order to get an insight into these results it is of advantage separately to consider the cases $l=0$ and $h=0$.

A. In the case $l=0$ (singular lines in a vertical row; compare fig. 12) we have: $\partial\Psi/\partial x=0$, $\partial\Psi/\partial y=0$; and consequently:

$$\left.\begin{aligned}&\text{for } x<0: \qquad u=u^*=-\frac{\lambda_0\, y}{2h}, \qquad v=v^*=+\frac{\lambda_0\, x}{2h}\\ &\text{for } x>0: \qquad u=+\frac{\lambda_0\, y}{2h}, \qquad v=-\frac{\lambda_0\, x}{2h}\end{aligned}\right\} \quad \ldots \quad (67)$$

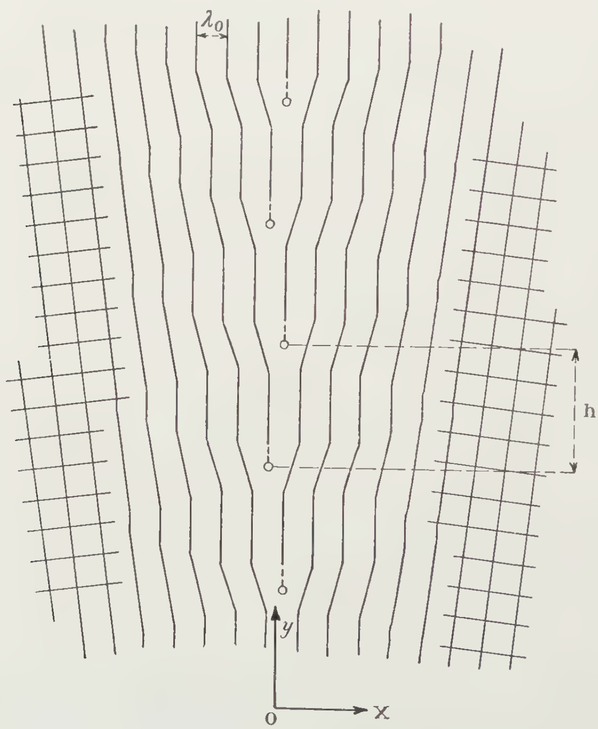

Fig. 12. Surface of misfit formed by parallel dislocation lines situated in the plane $x=0$.

These expressions make:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0. \quad \ldots \ldots \quad (68)$$

Hence they state that at large distances from the y, z-plane the lattices are inclined over the constant angle $+\lambda_0/2h$ on the left hand side, and $-\lambda_0/2h$ on the right hand side, without change of form. The deformations existing in the region near the y, z-plane completely disappear far away from this plane.

B. In the case $h = 0$ (singular lines in a horizontal row; see fig. 13) we have:

for $y > 0$:
$$u^* = -\frac{\lambda_0 x}{2l}; \qquad v^* = -\frac{\lambda_0 y}{2l}$$

$$\frac{\partial \Psi}{\partial x} = 0; \qquad \frac{\partial \Psi}{\partial y} = +\frac{\lambda + \mu}{\lambda + 2\mu} \frac{\lambda_0 y}{l}$$

and hence:

$$\left.\begin{aligned} & u = -\frac{\lambda_0 x}{2l}; \; v = +\frac{\lambda}{\lambda + 2\mu} \frac{\lambda_0 y}{2l} \\ \text{while for } y < 0: & \\ & u = +\frac{\lambda_0 x}{2l}; \; v = -\frac{\lambda}{\lambda + 2\mu} \frac{\lambda_0 y}{2l} \end{aligned}\right\} \quad \ldots \ldots \quad (69)$$

In this case above the row of singular lines there is a lateral compression,

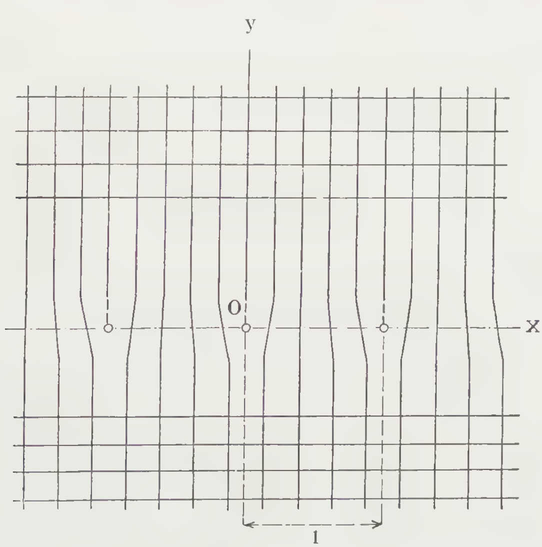

Fig. 13. Surface of misfit formed by parallel dislocation lines situated in the plane $y = 0$.

accompanied by an extension in the y-direction, while below the row the reverse situation is found.

The stress components become:

$$\left.\begin{array}{l} \text{for } y>0:\quad \sigma_{xx}=-\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\dfrac{\lambda_0}{l};\quad \sigma_{xy}=0;\quad \sigma_{yy}=0 \\ \text{and for } y<0: \\ \qquad\qquad \sigma_{xx}=+\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\dfrac{\lambda_0}{l};\quad \sigma_{xy}=0;\quad \sigma_{yy}=0 \end{array}\right\} \quad \ldots \ldots \quad (70)$$

The fact that we obtain a value for σ_{xx} which does not vanish at infinity, shows that in a block of finite extension in the x-direction the state of deformation described by our formulae can exist only if suitable pressures and tractions are applied at the lateral boundaries. When this is not the case, another deformation will be superposed upon the one calculated here, the precise nature of which will depend upon the form of the boundary.

23. The system obtained by taking $l=0$ (fig. 12) clearly is of similar nature as the case considered by TAYLOR and pictured schematically in fig. 2*a* (p. 392) of the paper mentioned in footnote 2) above. It seems plausible to suppose that the "surfaces of misfit" occurring in actual crystals give rise to lattice inclinations of small amount, generally less than a degree, and often measuring a few minutes of arc only. Such cases are obtained when the distance h between two consecutive dislocations is of the order of 100 λ_0 to 1000 λ_0. There is, however, a difference between the conception introduced here and TAYLOR's picture: TAYLOR appears to assume that the disturbance of the lattice in passing from one block to the other is small only in relatively small regions, represented in his fig. 2*a* mentioned above by the gaps in the line AB, where the distance of atoms on one side from the nearest atoms on the other is the same as that which belongs to the perfect crystal structure. In our picture on the other hand the regions where the two lattices are united in a regular way are of much larger extent than the regions where there is a disturbance; in particular when h is of the order 100 λ_0—1000 λ_0, the parts of the "surface of misfit" where there is an actual disturbance are of the order of a few percent only, perhaps even less, of the whole area. The two parts of the lattice in our picture are united so to say "in the best way possible" for a given angle of inclination between them. The "surface of misfit" in our conception therefore would be still less "opaque" than in TAYLOR's picture, so long as we restrict to the consideration of dislocations the singular lines of which are all parallel to the z-axis. Evidently the "surface of misfit" of our picture can be "opaque" to dislocations with singular lines in other directions, provided these lines are of sufficient length.

From the equations obtained in section **21** we can calculate the values

of σ_{xx} and σ_{xy} at the points of the "surface of misfit", *i.e.* of the y, z-plane. We have:

$$\sigma_{xx} = 2\,\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$\sigma_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).$$

As in fig. 12: $\xi = \pi\, y/h$, $\eta = -\pi\, x/h$, we find, for $x = 0$:

$$\left.\begin{aligned} &\frac{\partial u^*}{\partial x} = \frac{\partial v^*}{\partial y} = -\frac{\lambda_0}{2\,h} \cot \frac{\pi y}{h} \\ &\frac{\partial u^*}{\partial y} = -\frac{\partial v^*}{\partial\, x} \qquad \text{(everywhere in the field)} \end{aligned}\right\} \quad \ldots \ldots \quad (71)$$

$$\left.\begin{aligned} &\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Psi}{\partial y^2} = \frac{(\lambda + \mu)\, \lambda_0}{2\,(\lambda + 2\,\mu)\, h} \cot \frac{\pi y}{h} \\ &\frac{\partial^2 \Psi}{\partial x \partial y} = 0. \end{aligned}\right\} \quad \ldots \ldots \quad (72)$$

Hence in the plane $x = 0$:

$$\left.\begin{aligned} &\sigma_{xx} = -\frac{\mu\,(\lambda + \mu)\, \lambda_0}{(\lambda + 2\mu)\, h} \cot \frac{\pi y}{h} \\ &\sigma_{xy} = 0. \end{aligned}\right\} \quad \ldots \ldots \quad (73)$$

When x is different from zero we obtain:

$$\sigma_{xy} = 2\mu \frac{\partial^2 \Psi}{\partial x\, \partial y} = \frac{\mu\,(\lambda + \mu)\, \lambda_0}{(\lambda + 2\mu)\, h} \frac{\pi x}{h} \frac{(\cosh 2\,\pi x/h)\,(\cos 2\,\pi y/h) - 1}{2\,(\sinh^2 \pi x/h + \sin^2 \pi y/h)^2} \quad . \; . \quad (74)$$

These expressions can be used for making estimates of the magnitude of the stresses to be expected in given cases, in a similar way as is done by TAYLOR.

Other examples can be constructed, giving rise to a multitude of possible cases. For instance the "surface of misfit" can be repeated periodically at a distance L_0, so that a series of flat blocks is obtained. Or fields containing a finite number of dislocations may be investigated, and cases where the singular lines are not parallel straight lines, but lines of other form. However, it seems preferable for the present to leave the matter here.

Mathematics. — *Ueber fünf Erzeugende einer F_2 im R_4.* Von R. WEITZENBÖCK.

(Communicated at the meeting of March 25, 1939.)

Im dreidimensionalen projektiven Raume R_3 wird durch drei Geraden allgemeiner Lage eine Quadrik (= Fläche zweiter Ordnung als Punktort) F_2 bestimmt durch die Forderung, dass die drei Geraden Erzeugende der F_2 sein sollen.

Im vierdimensionalen Raume R_4 gehen durch vier gegebene Gerade als Erzeugende im Allgemeinen ∞^2 Quadriken F_2; denn die Forderung, dass eine Gerade eine Erzeugende sein soll, ist mit drei Bedingungen äquivalent. Da eine F_2 durch 15 Konstante bestimmt ist, müssen fünf Gerade allgemeiner Lage im R_4 einer Bedingung genügen, wenn sie Erzeugende der F_2 sein sollen. Es handelt sich im Folgenden um Aufstellung dieser Bedingung.

Zwei Geraden a_{ik} und α_{ik} bestimmen im R_4 einen Verbindungsraum S'_{12} mit der Gleichung

$$(xS'_{12}) = (xa^2\alpha^2) = 4 \,.\, \Sigma \pm x_1\, a_{23}\, \alpha_{45} = x_1\,(S'_{12})_1 + x_2\,(S'_{12})_2 + \ldots + x_5\,(S'_{12})_5 = 0. \quad (1)$$

Bei vier Geraden haben wir sechs solche Räume S'_{ik} und damit ergeben sich drei zerfallende Quadriken

$$F_{12,34} = (xS'_{12})\,(xS'_{34}) = 0,\; F_{13,42} = (xS'_{13})\,(xS'_{42}) = 0 \text{ und } F_{14,23} = (xS'_{14})\,(xS'_{23}) = 0,$$

die alle vier Geraden a, α, p und m enthalten. Dasselbe gilt für jede Quadrik des Büschels

$$F_\lambda = \lambda F_{12,34} + \mu F_{13,42} + \nu F_{14,23} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Man kann leicht beweisen, dass jede Quadrik, die die vier Geraden a, α, p und m enthält, in dieser Gestalt darstellbar ist. Ueberdies besteht der Schnitt aller F_λ aus acht Geraden: den vier genannten und den vier Transversalen durch je drei derselben.

Stellt man die Forderung, dass F_λ durch zwei gegebene Punkte y und z geht, so lassen sich λ, μ und ν aus den aus (2) folgenden Gleichungen eliminieren und man erhält als *Gleichung der Quadrik, die durch die vier Geraden a, α, p, m und durch die zwei Punkte y und z geht:*

$$Q_{xx} = \begin{vmatrix} (xS'_{12})\,(xS'_{34}) & (xS'_{13})\,(xS'_{42}) & (xS'_{14})\,(xS'_{23}) \\ (yS'_{12})\,(yS'_{34}) & (yS'_{13})\,(yS'_{42}) & (yS'_{14})\,(yS'_{23}) \\ (zS'_{12})\,(zS'_{34}) & (zS'_{13})\,(zS'_{42}) & (zS'_{14})\,(zS'_{23}) \end{vmatrix} = 0. \quad . \quad . \quad . \quad (3)$$

Hier ist $Q_{yy}=0$ und $Q_{zz}=0$. Fügen wir die Forderung $Q_{yz}=0$ hinzu, so liegt die Gerade yz ganz auf der Quadrik. Wir erhalten also

$$\triangle = \begin{vmatrix} (yS'_{12})(zS'_{34})+(zS'_{12})(yS'_{34}) & . & . \\ (yS'_{12})(yS'_{34}) & . & . \\ (zS'_{12})(zS'_{34}) & . & . \end{vmatrix} = 0. \quad . \quad . \quad . \quad (4)$$

Diese Gleichung lässt sich so umgestalten, dass in ihr die Koordinaten

$$\varphi_{ik}=(yz)_{ik}=y_i z_k - y_k z_i$$

der Geraden yz auftreten. Wir spalten $\triangle$ entsprechend den Elementen der ersten Zeile in zwei Determinanten: $\triangle=\triangle_1+\triangle_2$. Dabei entsteht $\triangle_2$ aus $\triangle_1$ durch Vertauschung von y mit z und Umkehrung des Zeichens. $\triangle_1$ entwickeln wir nach der letzten Zeile:

$$\triangle_1=(yS'_{12})(yS'_{14})(zy)_{42,23}(zS'_{13})(zS'_{34})+\ldots \quad . \quad . \quad . \quad (5)$$

Hier haben wir

$$(zy)_{42,23}=-(\varphi S'_{42})(\varphi S'_{23})=-\tfrac{1}{6}(S'_{42}\,S'_{23}\,\varphi'^3);$$

also wird der erste Term der rechten Seite von (5) gleich

$$-\tfrac{1}{6}(S'_{42}\,S'_{23}\,\varphi'^3)(S'_{34}\,z)(S'_{12}\,z)(yS'_{12})(yS'_{14}).$$

Hier formen wir das Produkt der ersten beiden Faktoren um:

$$(S'_{42}\,S'_{23}\,\varphi'^3)(S'_{34}\,z)=(S'_{34}\,S'_{23}\,\varphi'^3)(S'_{42}\,z)-(S'_{34}\,S'_{42}\,\varphi'^3)(S'_{23}\,z)+ \\ +3(S'_{34}\,S'_{42}\,S'_{23}\,\varphi'^2)(\varphi'\,z).$$

Der letzte Term verschwindet hier wegen des Faktors $(\varphi'\,z)$ und die beiden ersten Terme lassen sich mit den beiden ersten Gliedern der rechten Seite von (5) zusammenfassen. Man erhält schliesslich:

$$-36\,\triangle_1=(S'_{23}\,S'_{34}\,\varphi'^3)(yS'_{14})(zS'_{24})(S'_{13}\,S'_{12}\,\psi'^3)+ \\ +(S'_{34}\,S'_{42}\,\varphi'^3)(yS'_{13})(zS'_{23})(S'_{14}\,S'_{12}\,\psi'^3).$$

Ebenso bei $\triangle_2$; wenn wir φ als fünfte Gerade durch 5 andeuten, erhalten wir schliesslich

$$\triangle=A_{5,34,23}\,A_{5,13,12}\,A_{5,14,42}+A_{5,42,34}\,A_{5,14,12}\,A_{5,13,23}, \quad . \quad . \quad . \quad (6)$$

wobei z.B.

$$A_{5,34,23}=(\varphi S'_{34})(\varphi S'_{23})=(\varphi\,p^2\,m^2)(\varphi\,\alpha^2\,q^2)$$

u.s.w. gesetzt ist.

$\triangle=0$ *stellt die gesuchte Bedingung dar, dass fünf Geraden Erzeugende einer Quadrik im R_4 sind.* Sie ist in den Koordinaten jeder der drei Geraden vom dritten Grade. Die Geraden φ, die mit vier gegebenen zusammen fünf Erzeugende einer Quadrik sein können, bilden also einen Linienkomplex dritten Grades.

Mathematics. — *Sur quelques systèmes de congruences.* Par J. G. VAN DER CORPUT.

(Communicated at the meeting of March 25, 1939.)

Les recherches modernes de la théorie additive des nombres conduisent à l'étude suivante. Considérons une congruence de la forme

$$\psi(y) \equiv 0 \qquad (mod.\ p^\beta),$$

où $\psi(y)$ est un polynôme à coefficients entiers, p un nombre premier et β un nombre naturel. Désignons par Q_β le nombre des solutions de cette congruence. Il y a des polynômes pour lesquels Q_β croît indéfiniment avec β, par exemple la congruence

$$y^2 \equiv 0 \qquad (mod.\ p^\beta)$$

possède $p^{\frac{1}{2}\beta}$ ou $p^{\frac{1}{2}(\beta-1)}$ solutions, selon que β est pair ou impair. D'autre part il existe des polynômes $\psi(y)$ pour lesquels Q_β est borné. Il y a même des polynômes pour lesquels Q_β possède une même valeur pour tous les β supérieurs à une certaine borne. Ceci est le cas, comme M. LOO—KENG HUA [1]) l'a démontré, pour chaque polynôme $\psi(y)$ à coefficients entiers tel que le plus grand commun diviseur D de $\psi(y)$ et de sa dérivée $\psi'(y)$ soit indépendant de y. Dans ce cas M. HUA a démontré que Q_β possède la même valeur pour les $\beta \geqq 2\gamma+1$, où p^γ désigne la puissance la plus élevée de p qui divise D. Considérons maintenant un polynôme $\psi(y_1, \ldots, y_s)$ à coefficients entiers et désignons $\dfrac{\partial\psi}{\partial y_\sigma}$ par $\psi_\sigma(y_1, \ldots, y_s)$ $(\sigma = 1, \ldots, s)$. Supposons qu'il existe un nombre $\gamma \geqq 0$ tel que le système des s congruences

$$\psi(y_1, \ldots, y_s) \equiv 0\ (mod.\ p^{2\gamma+1});\ \psi_\sigma(y_1, \ldots, y_s) \equiv 0\ (mod.\ p^{\gamma+1}) \qquad (1)$$

$(\sigma = 1, \ldots, s)$ n'ait aucune solution. Si nous désignons par $p^{(s-1)\beta}\, Q_\beta$ le nombre des solutions de la congruence

$$\psi(y_1, \ldots, y_s) \equiv 0 \qquad (mod.\ p^\beta),$$

le nombre Q_β possède, comme nous le démontrerons, la même valeur pour tous les $\beta \geqq 2\gamma+1$.

La condition imposée est certainement remplie, si nous pouvons trouver $s+1$ polynômes à coefficients entiers $u(y_1, \ldots, y_s)$ et $u_\sigma(y_1, \ldots, y_s)$ tels que

$$u(y_1, \ldots, y_s)\, \psi(y_1, \ldots, y_s) + \sum_{\sigma=1}^{s} u_\sigma(y_1, \ldots, y_s)\, \psi_\sigma(y_1, \ldots, y_s)$$

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[1]) Journal of the London Mathematical Society, **13**, 54—61 (1938).

soit égal à un nombre $D \neq 0$ indépendant de $y_1, \ldots, y_s$. En effet, on peut alors choisir pour p^γ la puissance la plus élevée de p qui divise D. Ce fait, appliqué avec $s = 1$, donne le résultat cité plus haut de M. HUA.

Si $\psi(y_1, \ldots, y_s)$ est égal à $\Psi(y_1, \ldots, y_s) - t$, où $t \neq 0$ est indépendant de $y_1, \ldots, y_s$ et où $\Psi(y_1, \ldots, y_s)$ désigne une forme en $y_1, \ldots, y_s$ de degré $k \geqq 1$, on a

$$-k\,\psi(y_1, \ldots, y_s) - \sum_{\sigma=1}^{s} y_\sigma \psi_\sigma(y_1, \ldots, y_s) = k\,t;$$

dans ce cas la condition imposée est valable, si l'on choisit pour p^γ la puissance la plus élevée de p qui divise $k\,t$.

Généralisons le résultat précédent en considérant au lieu d'une congruence un système de congruences

$$\psi_\mu(y_1, \ldots, y_s) \equiv 0 \,(mod.\ p^\beta) \quad (\mu = 1, \ldots, m),$$

où s est $\geqq m$ et où $\psi_\mu(y_1, \ldots, y_s)$ désigne un polynôme à coefficients entiers. Comme nous le verrons, il est utile de considérer la matrice

$$M = \begin{pmatrix} \psi_{11} & \ldots & \psi_{1s} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \psi_{m1} & \ldots & \psi_{ms} \end{pmatrix},$$

où $\psi_{\mu\sigma} = \dfrac{\partial \psi_\mu}{\partial y_\sigma}$. Je supposerai que M possède le rang m. Les puissances $p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_m}$ s'appellent les diviseurs élémentaires modulo p de M, si $p^{\alpha_1 + \cdots + \alpha_\mu}$ est pour $\mu = 1, \ldots, m$ la puissance la plus élevée de p qui divise chaque déterminant d'ordre μ de M.

Ces diviseurs élémentaires modulo p ne changent pas par les transformations suivantes de M, qui s'appellent des transformations élémentaires [1]):

1. Echanger deux lignes entre elles ou deux colonnes entre elles.

2. Multiplier tous les éléments d'une ligne (colonne) par un même entier qui n'est pas divisible par p.

3. Ajouter aux éléments d'une ligne (colonne) les éléments correspondants d'une autre ligne (colonne) multipliés par un même entier.

4. Remplacer un élément par un entier qui lui est congru modulo $p^{\alpha_1 + \ldots + \alpha_m + 1}$.

Si une matrice M peut être transformée en une matrice M' par un nombre fini de transformations élémentaires, les matrices M et M' sont dites équivalentes.

Comme M renferme au moins un élément qui n'est pas divisible par p^{α_1+1}, il existe une matrice équivalente à M dont le premier élément n'est pas divisible par p^{α_1+1} (opération 1). Il existe donc une matrice équivalente à M dont le premier élément est égal à p^{α_1} (opérations 2 et 4).

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[1]) Comparez par exemple: M. BÔCHER, Introduction to Higher Algebra, 1924, Chapter XX, ou Einführung in die Höhere Algebra, 1910, Kapitel XX.

Comme tous les éléments de cette matrice sont divisibles par p^{α_1}, il existe une matrice équivalente à M, dont le premier élément est égal à p^{α_1} et dont les autres éléments figurant dans la première ligne ou première colonne s'annulent (opération 3). La répétition de ce raisonnement nous apprend que la matrice

$$M' = \begin{pmatrix} p^{\alpha_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & p^{\alpha_2} & \dots & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & p^{\alpha_m} & 0 & \dots & 0 \end{pmatrix}$$

est équivalente à M. Comme chaque élément de M' est divisible par p^{α_1}, on a $\alpha_1 \leqq \alpha_2$ et de la même manière on trouve $\alpha_1 \leqq \alpha_2 \leqq \dots \leqq \alpha_m$.

Si nous remplaçons dans M un élément par un entier qui lui est congru modulo p^{α_m+1}, les diviseurs élémentaires modulo p ne changent pas. En effet, un déterminant quelconque d'ordre μ de M est remplacé par un déterminant qui lui est congru modulo $p^{\alpha_1+\dots+\alpha_{\mu-1}+\alpha_m+1}$, donc modulo $p^{\alpha_1+\dots+\alpha_\mu+1}$.

Considérons maintenant le système de congruences

$$\sum_{\sigma=1}^{s} \psi_{\mu\sigma}(y_1, \dots, y_s)\, h_\sigma \equiv b_\mu\, p^{\alpha_m} \,(mod.\ p^\tau), \quad \dots \quad (3)$$

$(\mu = 1, \dots, m)$, où $b_1, \dots, b_m$ sont entiers et $\tau \geqq \alpha_m$. Les transformations 1, 2 et 3 de M transforment ce système de congruences en un système possédant le même nombre de solutions; au lieu des membres de droite on obtient les mêmes ou d'autres multiples de p^{α_m}. Le nombre des solutions de (3) ne change non plus, si $\psi_{\mu\sigma}(y_1, \dots, y_s)$ est remplacé par un entier qui lui est congru à $p^{\tau+1}$. Par cette opération et les opérations 1, 2 et 3 on peut transformer (3) en

$$p^{\alpha_\mu} h'_\mu \equiv b'_\mu\, p^{\alpha_m} \,(mod.\ p^\tau) \qquad (\mu = 1, \dots, m), \quad \dots \quad (4)$$

de façon que le nombre des solutions $(h_1, \dots, h_s)$ de (3) est égal au nombre des solutions $(h'_1, \dots, h'_s)$ de (4), donc égal à $p^{(s-m)\tau+\alpha_1+\dots+\alpha_m}$.

Passons après ces remarques préliminaires à la proposition 1.

Proposition 1: *Supposons qu'il existe un entier $\gamma \geqq 0$ tel que pour toute solution du système de congruences*

$$\psi_\mu(y_1, \dots, y_s) \equiv 0 \,(mod.\ p^{2\gamma+1}) \qquad (\mu = 1, \dots, m). \quad \dots \quad (5)$$

le rang de M soit m et que le dernier diviseur élémentaire modulo p de M soit $\leqq p^\gamma$. Si nous désignons par $p^{(s-m)\beta} Q_\beta$ le nombre des solutions de (2), *le nombre Q_β possède la même valeur pour chaque $\beta \geqq 2\gamma + 1$.*

On peut ajouter à cette proposition une autre proposition plus générale. Soit ν un entier $\geqq 0$. Je dis qu'une solution $y = (y_1, \dots, y_n)$ de (2) possède la propriété $P(\alpha_1, \dots, \alpha_m)$, si le rang de la matrice M est égal à m et

si $p^{\alpha_1}, \ldots, p^{\alpha_m}$ sont les diviseurs élémentaires modulo p de M. Comme nous le démontrerons, la proposition 1 découle immédiatement de la suivante.

Proposition 2: *Introduisons les entiers $s, m, \beta, w_1, \ldots, w_s$ tels que nous ayons $s \geqq m \geqq 1$ et $\beta \geqq 1$; introduisons en outre m polynômes $\psi_\mu(y_1, \ldots, y_s)$ à coefficients entiers et m entiers non-négatifs $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_m$. Posons $\alpha_m = \nu$ et $\beta \geqq 2\nu + 1$. Le nombre de solutions du système*

$$\psi_\mu(y_1, \ldots, y_s) \equiv 0 \ (mod.\ p^{\beta+1});\ y_\sigma \equiv w_\sigma \ (mod.\ p^{\beta-\nu}). \quad . \quad . \quad (6)$$

($\mu = 1, \ldots, m$; $\sigma = 1, \ldots, s$) possédant la propriété $P(\alpha_1, \ldots, \alpha_m)$, est exactement p^{s-m} multiplié par le nombre de solutions du système

$$\psi_\mu(y_1, \ldots, y_s) \equiv 0 \ (mod.\ p^{\beta});\ y_\sigma \equiv w_\sigma \ (mod.\ p^{\beta-\nu}) \quad . \quad . \quad . \quad (7)$$

avec cette même propriété.

Il découle de cette proposition que le système

$$\psi_\mu(y_1, \ldots, y_s) \equiv 0 \ (mod.\ p^{\beta+1}) \qquad (\mu = 1, \ldots, m) \quad . \quad . \quad . \quad (8)$$

possède exactement p^{s-m} fois autant de solutions avec la propriété $P(\alpha_1, \ldots, \alpha_m)$ que le système (2).

Il est facile de déduire la proposition 1 de la précédente. Il résulte des conditions de la proposition 1 que le système (2) ne possède aucune solution avec la propriété $P(\alpha_1, \ldots, \alpha_m)$, si $\beta \geqq 2\gamma + 1$ et $\alpha_m \geqq \gamma + 1$. On a par conséquent

$$Q_\beta = \sum_{\alpha_1, \ldots \alpha_m} Q_\beta(\alpha_1, \ldots, \alpha_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

où $p^{(s-m)\beta}\, Q_\beta(\alpha_1, \ldots, \alpha_m)$ désigne le nombre des solutions de (2) avec la propriété $P(\alpha_1, \ldots, \alpha_m)$. On obtient de la même manière

$$Q_{\beta+1} = \sum_{\alpha_1, \ldots \alpha_m} Q_{\beta+1}(\alpha_1, \ldots, \alpha_m),$$

où $p^{(s-m)(\beta+1)}\, Q_{\beta+1}(\alpha_1, \ldots, \alpha_m)$ désigne le nombre des solutions de (8) avec la propriété $P(\alpha_1, \ldots, \alpha_m)$. La proposition 2 nous apprend $Q_{\beta+1}(\alpha_1, \ldots, \alpha_m) = Q_\beta(\alpha_1, \ldots, \alpha_m)$, par conséquent $Q_{\beta+1} = Q_\beta$, si β est $\geqq 2\gamma + 1$.

Démonstration de la proposition 2.

Soit $y = (y_1, \ldots, y_s)$ une solution quelconque de (2) possédant la propriété $P(\alpha_1, \ldots, \alpha_m)$. Pour une solution $z = (z_1, \ldots, z_s)$ de (2) telle que

$$z_\sigma \equiv y_\sigma \ (mod.\ p^{\beta-\nu}) \qquad (\sigma = 1, \ldots, s), \quad . \quad . \quad . \quad . \quad . \quad (10)$$

la différence

$$\psi_{\mu\sigma}(z_1, \ldots, z_s) - \psi_{\mu\sigma}(y_1, \ldots, y_s)$$

est congrue à zéro, modulo $p^{\beta-\nu}$, donc aussi modulo $p^{\nu+1}$ (en vertu de $\beta - \nu \geqq \nu + 1$), de sorte que, d'après une remarque précédente, les diviseurs élémentaires modulo p de M ne changent pas, si l'on remplace y par z. La

solution z possède donc aussi la propriété $P(\alpha_1, \ldots, \alpha^m)$. Si j'appelle y et z deux solutions équivalentes, cette notion d'équivalence possède par conséquent les propriétés réflexive, symétrique et transitive.

Si y désigne une solution de (2), ayant la propriété $P(\alpha_1, \ldots, \alpha_m)$ et si l'on pose

$$z_\sigma = y_\sigma + g_\sigma p^{\beta-\nu} \qquad (\sigma = 1, \ldots s),$$

où g_σ est entier, on a

$$\psi_\mu(z_1, \ldots, z_s) \equiv \psi_\mu(y_1, \ldots, y_s) + \sum_{\sigma=1}^{s} g_\sigma p^{\beta-\nu} \psi_{\mu\sigma}(y_1, \ldots, y_s)$$

modulo $p^{2\beta-2\nu}$, donc aussi modulo p^β. On obtient par conséquent

$$\psi_\mu(z_1, \ldots, z_s) \equiv \sum_{\sigma=1}^{s} g_\sigma p^{\beta-\nu} \psi_{\mu\sigma}(y_1, \ldots, y_s) \, (mod.\ p^\beta).$$

Une condition nécessaire et suffisante pour que z soit une solution de (2) est donc

$$\sum_{\sigma=1}^{s} g_\sigma \psi_{\mu\sigma}(y_1, \ldots, y_s) \equiv 0 \ (mod.\ p^\nu).$$

D'après le raisonnement précédent (appliqué avec $\tau = \nu$) le nombre des solutions $g = (g_1, \ldots, g_s)$ de ce système est égal à $p^{(s-m+1)\nu+\omega}$, où $\omega = \alpha_1 + \ldots + \alpha_{m-1}$, de sorte qu'à chaque solution y de (2) avec la propriété $P(\alpha_1, \ldots, \alpha_m)$ correspondent exactement $p^{(s-m+1)\nu+\omega}$ solutions équivalentes, et ces solutions équivalentes possèdent toutes cette même propriété $P(\alpha_1, \ldots, \alpha_m)$. Une classe K de solutions de (7), équivalentes à une solution donnée qui possède la propriété $P(\alpha_1, \ldots, \alpha_m)$, est donc formée par $p^{(s-m+1)\nu+\omega}$ solutions de (7).

Si $u = (u_1, \ldots, u_s)$ est une solution de (8) avec la propriété $P(\alpha_1, \ldots, \alpha_m)$, toute solution $v = (v_1, \ldots, v_s)$ de (8) avec

$$v_\sigma \equiv u_\sigma \, (mod.\ p^{\beta+1-\nu}) \qquad (\sigma = 1, \ldots, s)$$

est dite équivalente à la solution u. Une classe K' de solutions de (6), équivalentes à une solution donnée qui possède la propriété $P(\alpha_1, \ldots, \alpha_m)$, est formée par $p^{(s-m+1)\nu+\omega}$ solutions de (6), d'après le résultat précédent, appliqué avec $\beta + 1$ au lieu de β. Toutes ces solutions de (6) possèdent la propriété $P(\alpha_1, \ldots, \alpha_m)$.

Soit r le nombre des classes différentes K des solutions de (7) qui possèdent la propriété $P(\alpha_1, \ldots, \alpha_m)$ et désignons par t le nombre des classes différentes K' de solutions de (6) qui possèdent cette propriété $P(\alpha_1, \ldots, \alpha_m)$. Comme chacune des classes K contient autant de solutions que chacune des classes K', il suffit de démontrer que t est égal à $p^{s-m} r$. Pour obtenir ce résultat je démontrerai qu'à chacune des classes K correspond bi-univoquement un système formé par p^{s-m} classes K'.

Soit K l'une quelconque des classes citées. Un élément arbitraire

$y=(y_1,\dots,y_s)$ de K est une solution de (7) avec la propriété $P(a_1,\dots,a_m)$. Si nous posons

$$u_\sigma = y_\sigma + h_\sigma\, p^{\beta-\nu} \qquad (\sigma=1,\dots,s), \quad . \quad . \quad . \quad . \quad . \quad (11)$$

nous avons pour $\mu=1,\dots,m$

$$\psi_\mu(u_1,\dots,u_s) \equiv \psi_\mu(y_1,\dots,y_s) + \sum_{\sigma=1}^{s} h_\sigma\, p^{\beta-\nu}\, \psi_{\mu\sigma}(y_1,\dots,y_s)$$

modulo $p^{2\beta-2\nu}$, par conséquent aussi modulo $p^{\beta+1}$ en vertu de $\beta \geqq 2\nu+1$. Une condition nécessaire et suffisante pour que $u=(u_1,\dots,u_s)$ soit une solution de (6) est donc

$$\sum_{\sigma=1}^{s} h_\sigma\, \psi_{\mu\sigma}(y_1,\dots,y_s) \equiv -p^{\nu} \cdot \frac{\psi_\mu(y_1,\dots,y_s)}{p^\beta} \,(mod.\ p^{\nu+1})$$

$(\mu=1,\dots,m)$ et, d'après le raisonnement précédent, ce système de congruences possède $p^{(s-m)(\nu+1)+\nu+\omega}$ solutions. De cette manière chaque solution $y=(y_1,\dots,y_s)$ de K fournit $p^{(s-m)(\nu+1)+\nu+\omega}$ solutions différentes u de (6), et chacune de ces solutions u possède la propriété $P(a_1,\dots,a_m)$, car $\psi_{\mu\sigma}(u_1,\dots,u_s)-\psi_{\mu\sigma}(y_1,\dots,y_s)$ est divisible par $p^{\beta-\nu}$, donc par $p^{\nu+1}$. Deux solutions différentes y et z de (7) appartenant à la même classe K donnent les mêmes solutions de (6), parce que y et z sont équivalents et vérifient donc les congruences (10). La classe K fournit ainsi exactement $p^{(s-m)(\nu+1)+\nu+\omega}$ solutions différentes de (6) qui toutes possèdent la propriété $P(a_1,\dots,a_m)$.

Si l'on trouve de cette manière une solution u de (6), on obtient aussi chaque solution de (6) qui est équivalente à u, de sorte qu'à la classe K correspondent

$$p^{(s-m)(\nu+1)+\nu+\omega} : p^{(s-m+1)\nu+\omega} = p^{s-m}$$

classes K'. Réciproquement, si une de ces classes K' est donnée, la classe correspondante K est définie univoquement; en effet, une solution quelconque u de (6) appartenant à K', est une solution de (7) ayant la propriété $P(a_1,\dots,a_m)$ et appartient à une classe K qui est ainsi fixée univoquement. La proposition 2 est donc démontrée.

Le corollaire suivant de cette proposition est utile dans la théorie additive des nombres. Considérons le système de congruences

$$\left.\begin{array}{lll} \chi_\mu(h_1,\dots,h_s) \equiv 0 \,(mod.\ p^\beta) & & (\mu=1,\dots,m) \\ h_\sigma \equiv u_\sigma & (mod.\ U_\sigma) & (\sigma=1,\dots,s), \end{array}\right\} \quad . \quad . \quad (12)$$

où $\chi_1,\dots,\chi_m$ désignent des polynômes à coefficients entiers, u_σ et U_σ des entiers donnés; U_σ est supposé positif. Je partagerai les nombres naturels $\sigma \leqq s$ en deux familles (l'une d'elles pouvant être vide) et je supposerai pour tout σ de la première famille que u_σ soit premier avec U_σ.

Désignons par $p^{(s-m)\beta} Q^*_\beta$ le nombre des systèmes $h = (h_1, \dots, h_s)$ formés par s nombres naturels $h_\sigma \leqq U_\sigma \, p^\beta$ $(\sigma = 1, \dots, s)$, qui vérifient le système (12) et remplissent en même temps la condition

$$\prod_\sigma{}' \, h_\sigma \not\equiv 0 \qquad (mod.\, p), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

où $\prod_\sigma'$ est étendu aux σ de la première famille; si la première famille est vide, la dernière condition est automatiquement remplie. Pour étudier ce nombre Q^*_β on peut, comme on le verra dans la proposition suivante, considérer la matrice

$$M^* = \begin{pmatrix} U_1 \chi_{11} & \dots & U_s \chi_{1s} \\ \cdot & \cdot \; \cdot \; \cdot \; \cdot & \cdot \\ U_1 \chi_{m1} & \dots & U_s \chi_{ms} \end{pmatrix},$$

où $\chi_{\mu\sigma} = \dfrac{\partial \chi_\mu}{\partial h_\sigma}$.

Proposition 3: *Supposons qu'il existe un entier $\gamma \geqq 0$ tel que le système*

$$\left.\begin{array}{ll} \chi_\mu (h_1, \dots, h_s) \equiv 0 \; (mod.\, p^{2\gamma+1}) & (\mu = 1, \dots, m) \\ h_\sigma \equiv u_\sigma \; (mod.\, U_\sigma) & (\sigma = 1, \dots, s) \end{array}\right\} \quad . \quad . \quad (14$$

ne possède aucune solution $h = (h_1, \dots, h_s)$ avec (13) et avec la propriété que le dernier diviseur élémentaire modulo p de la matrice M^* soit supérieur à p^γ.

Dans ces conditions Q^*_β possède la même valeur pour tout $\beta \geqq 2\gamma + 1$.

Démonstration: Si nous posons

$$\chi_\mu (u_1 + U_1 y_1, \dots, u_s + U_s y_s) = \psi_\mu (y_1, \dots, y_s),$$

$p^{(s-m)\beta} Q^*_\beta$ désigne le nombre des solutions du système

$$\psi_\mu (y_1, \dots, y_s) \equiv 0 \; (mod.\, p^\beta) \qquad (\mu = 1, \dots, m), \quad . \quad . \quad . \quad (15)$$

telles qu'on ait pour tout σ de la première famille

$$u_\sigma + U_\sigma \, y_\sigma \not\equiv 0 \qquad (mod.\, p). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Pour un σ de la première famille pour lequel U_σ est divisible par p, le nombre u_σ est premier avec U_σ, donc n'est pas divisible par p; pour un tel σ la congruence (16) est vérifiée d'elle même. Pour un σ de la première famille tel que U_σ ne soit pas divisible par p, il existe un seul nombre naturel $z_\sigma \leqq p$ avec la propriété que $u_\sigma + U_\sigma \, z_\sigma$ est divisible

par p. Par conséquent $p^{(s-m)\beta} Q^*_\beta$ est le nombre des solutions de (15) telles qu'on ait

$$y_\sigma \not\equiv z_\sigma \qquad (mod.\, p) \, . \, . \, . \, . \, . \, . \, . \, . \, . \qquad (17)$$

pour tout σ de la première famille avec $U_\sigma \not\equiv 0$ $(mod.\, p)$.

Supposons maintenant $\beta \geqq 2\gamma + 1$. Si une solution quelconque de (15) et (17) possède la propriété $P(\alpha_1, \ldots, \alpha_m)$, le nombre α_m est nécessairement $\leqq \gamma$. En effet, si α_m était $\geqq \gamma + 1$, le système $h = (h_1, \ldots, h_s)$ défini par

$$h_\sigma = u_\sigma + U_\sigma y_\sigma \qquad\qquad (\sigma = 1, \ldots, s)$$

serait une solution de (14) avec (13), telle que le dernier diviseur élémentaire modulo p de la matrice M^* serait supérieur à p^γ. On a donc

$$Q^*_\beta = \sum_{\alpha_1, \ldots, \alpha_m} Q_\beta(\alpha_1, \ldots, \alpha_m),$$

où $p^{(s-m)\beta} Q_\beta(\alpha_1, \ldots, \alpha_m)$ désigne le nombre des solutions de (15) et (17) possédant la propriété $P(\alpha_1, \ldots, \alpha_m)$. De la même manière on obtient

$$Q^*_{\beta+1} = \sum_{\alpha_1, \ldots \alpha_m} Q_{\beta+1}(\alpha_1, \ldots, \alpha_m)$$

où $p^{(s-m)(\beta+1)} Q_{\beta+1}(\alpha_1, \ldots, \alpha_m)$ désigne le nombre des solutions de (17) et

$$\psi_\mu(y_1, \ldots, y_m) \equiv 0 \, (mod.\, p^{\beta+1})$$

qui possèdent la propriété $P(\alpha_1, \ldots, \alpha_m)$. La proposition 2 nous apprend que $Q_\beta(\alpha_1, \ldots, \alpha_n)$ et $Q_{\beta+1}(\alpha_1, \ldots, \alpha_m)$, donc aussi Q^*_β et $Q^*_{\beta+1}$ possèdent les mêmes valeurs, si β est $\geqq 2\gamma + 1$.

Mathematics. — *Contribution à la théorie additive des nombres.* Par J. G. VAN DER CORPUT. (Sixième communication.)

(Communicated at the meeting of March 25, 1939.)

Dans cette communication [1]) je donnerai la démonstration de la proposition 12 (p. 8). Désignons par $c_{100}, c_{101}, \ldots, c_{116}$ des nombres, dépendant uniquement de M, ν, K, U, U', m et du choix du polynome $\psi(x)$, par C_7, C_8, C_9 et C_{10} des nombres dépendant uniquement de M, K, U, U', m et du choix du polynome $\psi(x)$. Comme je l'ai dit dans la communication précédente, cette démonstration est analogue à celle de la proposition 9 (p. 557—566).

Première partie de la démonstration:

Considérons dans cette partie de la démonstration les conditions [2]) de la proposition 1.

1. Je choisis pour V l'intervalle fermé $(2K, N)$, pour V' l'intervalle fermé (A', N) où $A' = N(\log N)^{-\frac{1}{2}gM}$ et pour T l'intervalle fermé $(2, N)$. Posons $r(v) = 1$ ou 0, selon que $\frac{v}{K}$ est un nombre premier $\equiv u \,(mod.\, U)$ ou non. Si nous posons en outre

$$\Gamma = \frac{1}{\log 2} \quad \text{et} \quad \varrho(v) = \frac{1}{K\varphi(U)\log\frac{v}{K}} (2K \leqq v \leqq N),$$

les inégalités (1) sont vérifiées.

Si N est suffisamment grand, chacun de ces trois intervalles V, V' et T renferme au moins un entier et au plus N entiers, A' est $\geqq 2b$ et à chaque $v' \geqq A'$ correspond un seul nombre positif x avec $\psi(x) = v'$.

Pour un entier v' $(A' \leqq v' \leqq N)$ auquel correspond au moins un nombre premier $p' \equiv u' \,(mod.\, U')$ tel que $\psi(p') = v'$, je pose $r'(v') = 1$ et pour les autres v' je pose $r'(v') = 0$. Je prendrai

$$\Gamma' = c_{100} A'^{-1+\frac{1}{g}}, \quad \ldots \ldots \ldots \quad (68)$$

où je fixerai c_{100} plus loin. Nous avons

$$\sum_{v} |r'(v')| \leqq \sum_{\substack{A' \leqq v' \leqq N \\ v' = \psi(x)}} 1 \leqq c_{101} N^{\frac{1}{g}} \leqq c_{101} N A'^{-1+\frac{1}{g}} \leqq \Gamma' N,$$

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[1]) Voir Proceedings Kon. Ned. Akad. v. Wetenschappen, A'dam, **41**, 227—237; 350—361; 442—453; 556—567 (1938); **42**, 2—12 (1939).

[2]) Voir Proceedings Kon. Ned. Akad. v. Wetenschappen, A'dam, **41**, 229 (1938).

si nous choisissons $c_{100} \geqq c_{101}$. Les nombres

$$\varrho'(v') = \frac{b^{\frac{1}{g}} v'^{-1+\frac{1}{g}}}{\varphi(U') \log \frac{v'}{b}} \qquad (A' \leqq v' \leqq N)$$

et

$$\sum_{A' \leqq v' \leqq N-1} |\varrho'(v+1) - \varrho'(v')|$$

sont $\leqq c_{102} A'^{-1+\frac{1}{g}} \leqq \Gamma'$, si l'on choisit $c_{100} \geqq c_{102}$.

La première condition de la proposition 1 est donc remplie.

2. Posons $l = 1$ et en outre pour les fractions irréductibles $\frac{a}{q}$ à dénominateur positif

$$\lambda\left(\frac{a}{q}\right) = \frac{\varphi(U)}{\varphi(qU)} \sum_{\substack{h=1 \\ h \equiv u \,(\mathrm{mod.}\, U) \\ (h,q)=1}}^{qU} e\left(\frac{aKh}{q}\right) \quad . \quad . \quad . \quad . \quad . \quad (69)$$

et

$$\lambda'\left(\frac{a}{q}\right) = \frac{\varphi(U')}{\varphi(qU')} \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,q)=1}}^{qU'} e\left(\frac{a\psi(h)}{q}\right) \quad . \quad . \quad . \quad . \quad . \quad (70)$$

où $e(\alpha) = e^{2\pi i \alpha}$. Les nombres $\lambda\left(\frac{a}{q}\right)$ et $\lambda'\left(\frac{a}{q}\right)$ sont en valeur absolue $\leqq q$, de sorte que (3) est valable, si l'on choisit $\gamma_1 \geqq 1$. L'expression

$$\sum_{\substack{A' \leqq v' \leqq N \\ v' \leqq y}} r'(v')\, e\left(\frac{av'}{q}\right) - \lambda'\left(\frac{a}{q}\right) \sum_{\substack{A' \leqq v' \leqq N \\ v' \leqq y}} \varrho'(v') =$$

$$\sum_{\substack{p' | qU' \\ A' \leqq \psi(p') \leqq N \\ p' \equiv u' \,(\mathrm{mod.}\, U') \\ \psi(p') \leqq y}} e\left(\frac{a\psi(p')}{q}\right) + \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h, qU')=1}}^{qU'} e\left(\frac{a\psi(h)}{q}\right) \left\{ \sum_{\substack{A' \leqq \psi(p') \leqq N \\ \psi(p') \leqq y \\ p' \equiv h \,(\mathrm{mod.}\, qU')}} 1 - \frac{b^{-\frac{1}{g}}}{\varphi(qU')} \sum_{\substack{A' \leqq v' \leqq N \\ v' \leqq y}} \frac{v'^{-1+\frac{1}{g}}}{\log \frac{v'}{b}} \right\}$$

est en valeur absolue d'après de lemme 14 (appliqué avec $k = qU'$) pour tout nombre naturel m inférieure à

$$qU' + q C_7 N^{\frac{1}{g}} (\log N)^{-m} \leqq C_8\, q\, N A'^{-1+\frac{1}{g}} (\log N)^{-m}.$$

En remplacant $\psi(x)$ par Kx, par conséquent g par 1, on obtient donc que

$$\sum_{\substack{2 < v \leqq N \\ v < y}} r(v)\, e\left(\frac{av}{q}\right) - \lambda\left(\frac{a}{q}\right) \sum_{\substack{2 < v \leqq N \\ v \leqq y}} \varrho(v)$$

est en valeur absolue inférieure à

$$q\,U + q\,C_9\,N(\log N)^{-m} \leqq C_{10}\,q\,N(\log N)^{-m}.$$

Par suite les inégalités (4) et (5) sont valables, si l'on choisit

$$\gamma_m \geqq C_{10}, \quad \gamma_m \geqq C_8 \quad \text{et} \quad c_{100} \geqq 1.$$

3. A chaque $v' \geqq A'$ correspond un seul $x > 0$ tel que $\psi(x) = v'$. Si nous introduisons les nombres positifs α' et β' avec les propriétés $\psi(\alpha') = A'$ et $\psi(\beta') = N$, nous avons

$$\sum_{A' \leqq v \leqq N} r'(v')\, e^{2\pi i \alpha v'} = \sum_{\substack{\alpha' \leqq p' \leqq \beta' \\ p' \equiv u' \,(\mathrm{mod.}\, U')}} e^{2\pi i \alpha \psi(p')} \quad . \quad . \quad . \quad . \quad (71)$$

Posons $\eta_m = 1 + \zeta$, où ζ désigne le nombre figurant dans le lemme 19 (These Proceedings, p. 11). Si N est suffisamment grand, on a

$$N^{-1}(\log N)^{\eta_m} \geqq \beta'^{-g}(\log \beta')^{\zeta} \quad \text{et} \quad (\log N)^{\eta_m} \geqq (\log \beta')^{\zeta}.$$

Pour tout nombre réel α tel que l'intervalle fermé $(\alpha \mp N^{-1}(\log N)^{\eta_m})$ ne contienne aucune fraction à dénominateur $\leqq (\log N)^{\eta_m}$, l'intervalle $(\alpha \mp \beta'^{-g}(\log \beta')^{\zeta})$ ne contient aucune fraction à dénominateur $\leqq (\log \beta')^{\zeta}$ et le lemme 19 nous apprend donc que le membre de droite de (6) est en valeur absolue

$$\leqq c_{103}\,\beta'(\log \beta')^{-m} \leqq c_{104}\,N^{\frac{1}{g}}(\log N)^{-m} \leqq c_{104}\,N A'^{-1+\frac{1}{g}}(\log N)^{-m}$$
$$\leqq \gamma_m\,\Gamma'\,N(\log N)^{-m},$$

si nous choisissons $\gamma_m \geqq c_{104}$ et $c_{100} \geqq 1$.

Ainsi nous avons démontré que les conditions de la proposition 1 sont remplies.

Deuxième proposition de la démonstration:

Considérons maintenant les conditions de la propositions 5. (Proceedings, 41, p. 443). Pour démontrer que la fonction

$$H(q, t) = \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \lambda\left(\frac{a}{q}\right) \lambda'\left(\frac{a}{q}\right) e^{-\frac{2\pi i a t}{q}} \quad . \quad . \quad . \quad . \quad (72)$$

possède, pour toute paire de nombres naturels q_1 et q_2 qui sont premiers entre eux, la propriété multiplicative $H(q_1, t)\, H(q_2, t) = H(q_1 q_2, t)$, il

suffit, comme nous l'avons vu dans la deuxième communication (p. 354) de déduire les relations

$$\lambda\begin{pmatrix} a_1 \\ q_1 \end{pmatrix}\lambda\begin{pmatrix} a_2 \\ q_2 \end{pmatrix}=\lambda\begin{pmatrix} a_1 & + & a_2 \\ q_1 & & q_2 \end{pmatrix} \text{ et } \lambda'\begin{pmatrix} a_1 \\ q_1 \end{pmatrix}\lambda'\begin{pmatrix} a_2 \\ q_2 \end{pmatrix}=\lambda'\begin{pmatrix} a_1 & + & a_2 \\ q_1 & & q_2 \end{pmatrix}$$

pour un couple quelconque de fractions irréductibles $\frac{a_1}{q_1}$ et $\frac{a_2}{q_2}$ dont les dénominateurs q_1 et q_2 sont premiers entre eux. Or, ces relations découlent immédiatement du lemme 13 (p. 556) en posant $\chi(x)=Kx$ ou $\chi(x)=\psi(x)$.

Posons pour tout nombre premier p et pour tout entier $\sigma\geqq 0$

$$T_\sigma=\frac{p^\sigma\,\varphi(U')}{\varphi(p^\sigma\,U')}\sum_{\substack{h=1\\ h\equiv u'\,(mod.\,U')\\ (h,p^\sigma)=1\\ p^\sigma/\psi(h)-t}}^{p^\sigma U'} 1\,; \quad \ldots\ldots\ldots \quad (73)$$

T_0 est donc égal à 1. Je dis que pour chaque nombre naturel σ

$$\frac{p^{\sigma-1}\,\varphi(U')}{\varphi(p^\sigma\,U')}\sum_{\substack{h=1\\ h\equiv u'\,(mod.\,U')\\ (h,p)=1\\ p^{\sigma-1}/\psi(h)-t}}^{p^\sigma U'} 1=T_{\sigma-1}. \quad \ldots\ldots \quad (74)$$

En effet, dans le cas où σ est égal à 1 et p n'est pas un facteur de U', le premier membre de (74) égale

$$\frac{\varphi(U')}{\varphi(p\,U')}\cdot(p-1)=1=T_0.$$

Dans les autres cas à tout nombre naturel $k\leqq p^{\sigma-1}\,U'$ correspondent précisément p nombres naturels $h\leqq p^\sigma\,U'$ définis par

$$h\equiv k \qquad (mod.\ p^{\sigma-1}\,U')\,;$$

les relations

$$k\equiv u'\,(mod.\,U')\,; \quad (k,p^{\sigma-1})=1\,; \quad p^{\sigma-1}/\psi(k)-t$$

entraînent

$$h\equiv u'\,(mod.\,U')\,; \quad (h,p)=1\,; \quad p^{\sigma-1}/\psi(h)-t$$

et réciproquement (dans le cas $\sigma=1$, p est un facteur de U', de sorte que les relations $h\equiv u'$ $(mod.\,U')$ et $(u',U')=1$ impliquent $(h,p)=1$); le membre de gauche de (74) est donc égal à

$$\frac{p^{\sigma-1}\,\varphi(U')}{\varphi(p^\sigma\,U')}\cdot p\sum_{\substack{k=1\\ k\equiv u'\,(mod.\,U')\\ (k,p^{\sigma-1})=1\\ p^{\sigma-1}/\psi(k)-t}}^{p^{\sigma-1} U'} 1=T_{\sigma-1}.$$

Dans l'évaluation de

$$H(p^\sigma, t) = \frac{\varphi(U')}{\varphi(p^\sigma U')} \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p)=1}}^{p^\sigma U'} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma - 1} \lambda\left(\frac{a}{p^\sigma}\right) e\left(\frac{a}{p^\sigma}(\psi(h) - t)\right)$$

$(\sigma \geqq 1)$ (comparez (70) et (72)), je distinguerai cinq cas différents.

1. Soit $p \nmid U$ et p^σ/K. La relation (69) nous apprend $\lambda\left(\frac{a}{p^\sigma}\right) = 1$, par suite

$$\left.\begin{aligned} H(p^\sigma, t) &= \frac{\varphi(U')}{\varphi(p^\sigma U')} \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p)=1}}^{p^\sigma U'} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma - 1} e\left(\frac{a}{p^\sigma}(\psi(h) - t)\right) \\ &= \frac{\varphi(U')}{\varphi(p^\sigma U')} \Bigg\{ \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p)=1 \\ p^\sigma / \psi(h) - t}}^{p^\sigma U'} (p^\sigma - p^{\sigma-1}) - \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p)=1 \\ p^\sigma \nmid \psi(h) - t \\ p^{\sigma-1} / \psi(h) - t}}^{p^\sigma U'} p^{\sigma-1} \Bigg\} \\ &= T_\sigma - T_{\sigma-1} \end{aligned}\right\} \quad \ldots \tag{75}$$

en vertu de (73) et (74).

2. Soit $p \nmid U$; $p^\sigma \nmid K$; $p^{\sigma-1}/K$. Le nombre $\lambda\left(\frac{a}{p^\sigma}\right)$, defini par (69), est égal à $-\frac{\varphi(U)\, p^{\sigma-1}}{\varphi(p^\sigma U)} = \frac{-1}{p-1}$, de sorte qu'on obtient

$$H(p^\sigma, t) = -\frac{T_\sigma - T_{\sigma-1}}{p-1}.$$

3. Dans le cas où $p \nmid U$ et $p^{\sigma-1} \nmid K$, les nombres $\lambda\left(\frac{a}{p^\sigma}\right)$ et $H(p^\sigma, t)$ s'annulent.

4. Soit p/U et p^σ/KU. On a

$$\begin{aligned} H(p^\sigma, t) &= \frac{\varphi(U')}{\varphi(p^\sigma U')} \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p)=1}}^{p^\sigma U'} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^\sigma - 1} e\left(\frac{a}{p^\sigma}(\psi(h) + Ku - t)\right) \\ &= T'_\sigma - T'_{\sigma-1}, \end{aligned}$$

si l'on pose pour $\sigma \geqq 0$

$$T'_\sigma = \frac{p^\sigma \varphi(U')}{\varphi(p^\sigma U')} \sum_{\substack{h=1 \\ h \equiv u' \,(\mathrm{mod.}\, U') \\ (h,p^\sigma)=1 \\ p^\sigma / \psi(h) + Ku - t}}^{p^\sigma U'} 1.$$

5. Dans le cas où p/U et $p^\sigma \dagger KU$ les nombres $\lambda\left(\frac{a}{p^\sigma}\right)$ et $H(p^\sigma, t)$ s'annulent.

Démontrons maintenant qu'il existe un nombre γ dépendant uniquement de K, U, U', G et g tel qu'on ait

$$|H(p^\sigma, t)| < \frac{\gamma}{p^\sigma} \quad \ldots\ldots\ldots \quad (76)$$

pour tout nombre premier p, pour tout entier t et pour tout nombre naturel σ. Comme nous le savons, G désigne le nombre défini au commencement de la cinquième communication (p. 2), c. à d. G est le produit de tous les nombres premiers P qui ne sont pas un facteur de U' et pour lesquels les congruences

$$\psi(x) \equiv \psi(1) \qquad (mod.\ P)$$

sont valables pour tous les entiers x non divisibles par P. L'inégalité (76) est évidente, si p^σ/KU et également si $\sigma \geqq 2$ et $p^{\sigma-1}/K$. Elle est aussi évidente si $H(p^\sigma, t) = 0$. D'après le raisonnement précédent seulement le cas $\sigma = 1$ et $p \dagger KU$ reste. Je puis supposer $p \dagger U'$; sinon (76) avec $\sigma = 1$ est évident. Si tous les coefficients du polynome $\psi(x) - \psi(1)$ sont divisibles par p, ce nombre p est un facteur de G et (76) est immédiat. Si au moins un des coefficients du polynome $\psi(x) - \psi(1)$ n'est pas divisible par p, la congruence $\psi(h) - t \equiv 0\ (mod.\ p)$ possède tout au plus g solutions et

$$T_1 \leqq \frac{p\varphi(U')}{\varphi(pU')} \sum_{\substack{h=1 \\ p/\psi(h)-t}}^{pU'} 1 \leqq \frac{p}{p-1} U' g,$$

de sorte que

$$H(p, t) = -\frac{T_1 - 1}{p-1}$$

est en valeur absolue inférieur à $\frac{\gamma}{p}$, où γ désigne un nombre convenablement choisi. Ainsi (76) est démontré.

D'après la propriété multiplicative de la fonction $H(q, (t)$ on a pour tout nombre naturel q et pour tout entier t

$$H(q, t) = \prod_{p^\sigma/q} H(p^\sigma, t) \leqq \prod_{\substack{p^\sigma/q \\ p^{\frac{\sigma}{m}} < \gamma}} \frac{\gamma}{p^\sigma} \cdot \prod_{\substack{p^\sigma/q \\ p^{\frac{\sigma}{m}} \geqq \gamma}} \frac{1}{p^{\sigma\left(1-\frac{1}{m}\right)}} < \frac{c_{105}}{q^{1-\frac{1}{m}}}.$$

Par conséquent les conditions de la proposition 5 (p. 443) sont remplies.

Troisiéme partie de la démonstration:

Si nous posons

$$L(t)=\sum_{v+v'=t} r(v)\,r'(v'); \qquad \Lambda(t)=\sum_{v+v'=t} \varrho(v)\,\varrho'(v')$$

et

$$\Omega_\nu(t)=\prod_p \Bigl(1+\sum_{\sigma=1}^{\left[\frac{\nu \log\log t}{\log p}\right]} H(p^\sigma, t)\Bigr),$$

la proposition 5, appliquée avec $m=g\,M$ (ν est donc $>m$) nous apprend

$$\left.\begin{aligned}\sum_{t=2}^{[N]} |L(t)-\Lambda(t)\,\Omega_\nu(t)|^2 &< c_{106}\,\Gamma^2\,\Gamma'^2\,N^3\,(\log N)^{-m}\\ &< c_{107}\,A'^{-2+\frac{2}{g}}\,N^3\,(\log N)^{-gM}\\ &< c_{108}\,N^{1+\frac{2}{g}}\,(\log N)^{-gM+\frac{1}{2}gM\left(2-\frac{2}{g}\right)}\\ &= c_{108}\,N^{1+\frac{2}{g}}\,(\log N)^{-M},\end{aligned}\right\} \tag{77}$$

en vertu de (68).

Soit p un nombre premier quelconque et désignons par p^ω la puissance la plus élevée de p qui est un diviseur de KU. Si p n'est pas un facteur de U, on a

$$\begin{aligned}1+\sum_{\sigma=1}^{\infty} H(p^\sigma, t) &= 1+\sum_{\sigma=1}^{\omega+1} H(p^\sigma, t)\\ &= 1+\sum_{\sigma=1}^{\omega} (T_\sigma - T_{\sigma-1}) - \frac{T_{\omega+1}-T_\omega}{p-1}\\ &= T_\omega - \frac{T_{\omega+1}}{p-1}+\frac{T_\omega}{p-1}=\frac{p\,T_\omega - T_{\omega+1}}{p-1}\\ &= \frac{p^{\omega+1}\,\varphi(U')}{(p-1)\,\varphi(p^{\omega+1}\,U')}\Bigl\{\sum_{\substack{h=1\\ h\equiv u'\,(\mathrm{mod.}\,U')\\ (h,p)=1\\ p^\omega \mid \psi(h)-t}}^{p^{\omega+1}U'} 1 - \sum_{\substack{h=1\\ h\equiv u'\,(\mathrm{mod.}\,U')\\ (h,p)=1\\ p^{\omega+1} \mid \psi(h)-t}}^{p^{\omega+1}U'} 1\Bigr\}\\ &= W(p,t)\end{aligned}$$

d'après la définition de cette dernière fonction (voir p. 6). Si par contre p est un facteur de U, on a

$$\begin{aligned}1+\sum_{\sigma=1}^{\infty} H(p^\sigma, t) &= 1+\sum_{\sigma=1}^{\omega} H(p^\sigma, t) = 1+\sum_{\sigma=1}^{\omega} (T'_\sigma - T'_{\sigma-1}) = T'_\omega\\ &= \frac{p^\omega\,\varphi(U')}{\varphi(p^\omega\,U')}\sum_{\substack{h=1\\ h\equiv u'\,(\mathrm{mod.}\,U')\\ (h,p^\omega)=1\\ p^\omega \mid \psi(h)+Ku-t}}^{p^\omega U'} 1 = W(p,t).\end{aligned}$$

Supposons maintenant que t soit >1 et que $(\log t)^{\nu}$ soit $\geqq K^2 U^2$. Pour un facteur premier p de KU chaque nombre $\sigma > \left[\frac{\nu \log \log t}{\log p}\right]$ satisfait à

$$p^{\sigma} > (\log t)^{\nu} \geqq K^2 U^2,$$

d'où il suit $\sigma > 2\omega \geqq \omega + 1$, de sorte que $H(p^{\sigma}, t)$ s'annule. Pour un nombre premier $p \leqq (\log t)^{\nu}$ qui n'est pas un facteur de KU on a $\omega = 0$, de sorte que $H(p^{\sigma}, t)$ s'annule pour tout entier $\sigma > 1$, par suite pour tout entier $\sigma > \left[\frac{\nu \log \log t}{\log p}\right]$. Ainsi nous trouvons

$$\Omega_{\nu}(t) = \prod_{p} \Bigl(1 + \sum_{\sigma=1}^{\left[\frac{\nu \log \log t}{\log p}\right]} H(p^{\sigma}, t)\Bigr) = \prod_{p \leqq (\log t)^{\nu}} \Bigl(1 + \sum_{\sigma=1}^{\infty} H(p^{\sigma}, t)\Bigr)$$

$$= \prod_{p \leqq (\log t)^{\nu}} W(p, t) = \Omega_{\nu}^{*}(t).$$

en vertu de (62). L'inégalité (77) nous apprend donc

$$\sum_{t=2}^{[N]} |L(t) - \Lambda(t)\, \Omega_{\nu}^{*}(t)|^2 < c_{109} N^{1+\frac{2}{g}} (\log N)^{M}, \quad . \quad . \quad . \quad (78)$$

puisque les termes dans lesquels $(\log t)^{\nu}$ est inférieur à $K^2 U^2$ fournissent une contribution dont la valeur absolue est $< c_{110}$.

Maintenant nous sommes presque prêts, car il suffit de démontrer qui'il est permis de remplacer dans la dernière inégalité $L(t)$ per $F(t)$ et $\Lambda(t)$ par $\Phi(t)$, si nous remplaçons dans le nombre de droite c_{109} par un autre nombre dépendant uniquement de M, ν, K, U, U' et du choix du polynome $\psi(x)$.

Considérons d'abord $W(p, t)$ où le nombre premier p n'est pas un facteur de $KUU'G$. D'après la définition (p. 6) $p - \frac{(p-1)^2}{p} W(p, t)$ est le nombre des nombres naturels h tels que

$$h \leqq p U'; \qquad h \equiv u' \ (mod.\ U'); \qquad p / h(\psi(h) - t).$$

Il est impossible que tous les coefficients du polynome

$$\Psi(X) = (u' + U' X)(\psi(u' + U' X) - t)$$

sont divisible par p. En effet, dans ce cas on aurait pour tout nombre $y \equiv u' \ (mod.\ U')$

$$y(\psi(y) - t) \equiv 0 \qquad (mod.\ p);$$

à chaque entier x correspondrait un nombre $y \equiv u'$ (*mod.* U') avec $y \equiv x$ (*mod.* p) et on trouverait pour tout entier x

$$x(\psi(x)-t) \equiv 0 \qquad (\textit{mod. } p),$$

par suite pour tout entier x non divisible par p

$$\psi(x) \equiv t; \qquad \psi(1) \equiv t; \qquad \psi(x) \equiv \psi(1) \quad (\textit{mod. } p),$$

de sorte que p serait un facteur de G, ce qui est contraire à l'hypothèse. De cette manière nous avous trouvé que les coefficients du polynome $\Psi(x)$ ne sont pas tous divisibles par p, de sorte que la congruence $\Psi(x) \equiv 0$ (*mod.* p) possède tout au plus $g+1$ solutions. Par suite on a

$$p - \frac{(p-1)^2}{p} W(p, t) \geqq 0 \text{ et } \leqq g+1,$$

d'où il découle

$$|W(p, t)-1| \leqq \frac{2g}{p-1}.$$

Le lemme 12 (p. 452) nous apprend pour tout entier $t > e^2$

$$|\Omega_\nu^*(t)| \leqq c_{111} (\log\log t)^{2g}.$$

Traitons ensuite $L(t)$, le nombre des nombres premiers $p \equiv u$ (*mod.* U) avec $p \geqq 2K$ tel que $t - Kp$ soit égal à $\psi(p')$, où p' désigne un nombre premier $\equiv u'$ (*mod.* U') avec $\psi(p') \geqq A'$. On a donc

$$0 \leqq F(t) - L(t) \leqq c_{112} A'^{\frac{1}{g}}.$$

Considérons enfin

$$\Lambda(t) = c_{112} \sum_{\substack{v \geqq 2K \\ v' \geqq A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\left(\log \frac{v}{K}\right)\left(\log \frac{v'}{b}\right)}$$

où

$$c_{112}^{-1} = K b^{\frac{1}{g}} \varphi(U) \varphi(U').$$

Nous avons

$$0 \leqq \Phi(t) - \Lambda(t) = c_{112} \sum_{\substack{v \geqq 2K \\ 2b \leqq v' < A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\left(\log \frac{v}{K}\right)\left(\log \frac{v'}{b}\right)}$$

$$\leqq \frac{c_{112}}{\log 2} \sum_{\substack{v \geqq 2K \\ 2 \leqq v' < A' \\ v+v'=t}} \frac{v'^{-1+\frac{1}{g}}}{\log \frac{v}{K}} < \frac{c_{113} A'^{\frac{1}{g}}}{\log t}$$

d'après la démonstration donnée à la fin de la quatrième communication (p. 566). Nous obtenons donc

$$| F(t) - \Phi(t)\, \Omega_\nu^*(t) | < | L(t) - \Lambda(t)\, \Omega_\nu^*(t) | + c_{114} A'^{\frac{1}{g}},$$

d'où il suit en vertu de (78)

$$\sum_{t=2}^{[N]} | F(t) - \Phi(t)\, \Omega_\nu^*(t) |^2 < 2 c_{109} N^{1+\frac{2}{g}} (log\, N)^{-M} + c_{115} N A'^{\frac{2}{g}}$$

$$= c_{116} N^{1+\frac{2}{g}} (log\, N)^{-M}.$$

Ainsi la proposition 12 est démontrée.

Bien que nous n'ayons pas encore démontré ainsi tous les résultats annoncés dans l'introduction, nous arrêterons ici cette série de communications, parce que sous peu nous ferons paraître dans les Acta Arithmetica trois théorèmes plus généraux et plus aisés à appliquer que ceux que nous avons obtenus dans les notes précédentes.

Mathematics. — *Sur un problème de* WARING *généralisé.* I. Par J. G. VAN DER CORPUT et CH. PISOT.

(Communicated at the meeting of March 25, 1939.)

§ 1. *Introduction.*

Dans trois récents théorèmes VAN DER CORPUT [1]) a étudié la représentation d'un système $t = (t_1, \ldots, t_m)$ de m entiers t_μ par des expressions de la forme

$$t_\mu = \sum_{\nu=1}^{n} b_{\mu\nu}\, v_\nu \qquad (\mu = 1, 2, \ldots, m) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

où $b_{\mu\nu}$ sont des entiers donnés et v_ν des entiers possédant certaines propriétés. De ces trois théorèmes qu'il a appelés A, B et C, nous utiliserons dans cette communication seulement le théorème A. Nous nous bornerons au cas où les v_ν sont des polynomes $f_\nu(y_\nu)$ à coefficients entiers dont le degré exact est $k_\nu \geqq 2$; y_ν est un nombre naturel tel que $|f_\nu(y_\nu)| \leqq X$, où X désigne un entier quelconque supérieur à 2. (Dans une communication ultérieure nous nous proposons aussi d'étudier le cas où les y_ν sont des nombres premiers.)

Le théorème A donne sous certaines conditions une valeur approximative du nombre $L(t)$ de représentations du système t sous la forme indiquée. Cette valeur approximative se présente comme un produit $b\,Z(t)\,\Lambda(t)$ où b^{-1} est le nombre de systèmes $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ de nombres rationnels γ_μ avec $0 \leqq \gamma_\mu < 1$ tels que les n nombres $\sum_{\mu=1}^{n} b_{\mu\nu}\,\gamma_\mu$ soient entiers. Pour définir $\Lambda(t)$ nous remarquons qu'il existe une constante c_1 telle que pour tout nombre réel $\eta_\nu \geqq c_1$, chaque dérivée $f'_\nu(\eta_\nu) \neq 0$. Quand η_ν parcourt toutes les valeurs supérieures ou égales à c_1, le nombre $v_\nu = f_\nu(\eta_\nu)$ parcourt un intervalle j_ν et la correspondance entre un $\eta_\nu \geqq c_1$ et un v_ν de j_ν est biunivoque.

Alors $\Lambda(t)$ est la somme

$$\Sigma \frac{1}{|f'_1(\eta_1) \ldots f'_n(\eta_n)|}.$$

étendue aux entiers v_1 de $j_1, \ldots, v_n$ de j_n tels que l'on ait

$$\sum_{\nu=1}^{n} b_{\mu\nu}\, v_\nu = t_\mu \quad (\mu = 1, \ldots, m) \quad \text{et} \quad |v_\nu| \leqq X \quad (\nu = 1, \ldots, n).$$

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[1]) VAN DER CORPUT, Propriétés additives, Acta arithmetica (sous presse).

Enfin $Z(t)$, appelé par VAN DER CORPUT „facteur arithmétique", dépend du caractère arithmétique des nombres t_μ.

On démontre aisément que l'hypothèse E figurant dans le théorème A est vérifiée.

Pour certaines applications il est nécessaire de soumettre le facteur arithmétique à un examen plus détaillé. C'est l'objet du § 3 dû à VAN DER CORPUT. Mais avant de pouvoir appliquer le résultat du § 3, il faut démontrer que, dans certaines conditions, le nombre de solutions de deux certains systèmes n'est pas trop grand. Soit l un nombre naturel inférieur à n et soit N le nombre de solutions [2]) $y_1, \ldots, y_l, z_1, \ldots, z_l$ du système:

$$\left.\begin{array}{ll} \sum_{\lambda=1}^{l} b_{\mu\lambda}(f_\lambda(y_\lambda) - f_\lambda(z_\lambda)) = 0 & (\mu = 1, 2, \ldots, m) \\ |f_\lambda(y_\lambda)| \leqq X, \quad |f_\lambda(z_\lambda)| \leqq X & (\lambda = 1, 2, \ldots, l) \end{array}\right\} \quad \ldots \quad (2)$$

Pour pouvoir appliquer le théorème A, il est nécessaire de démontrer pour tout nombre positif fixe ε que

$$N \leqq C_1 X^{2\left(\frac{1}{k_1}+\ldots+\frac{1}{k_l}\right)-m+\varepsilon} \quad \ldots \ldots \ldots \quad (3)$$

Dans cette communication les lettres C désignent toujours des constantes convenables dépendantes de ε, mais indépendantes de X.

En outre on a besoin d'une borne supérieure analogue pour le nombre N^* de solutions du système:

$$\left.\begin{array}{ll} \sum_{\lambda=l+1}^{n-1} b_{\mu\lambda}(f_\lambda(y_\lambda) - f_\lambda(z_\lambda)) & (\mu = 1, 2, \ldots, m) \\ f_\lambda(y_\lambda) \leqq X, \quad f_\lambda(z_\lambda) \leqq X & (\lambda = l+1, \ldots, n-1) \end{array}\right\} \quad \ldots \quad (4)$$

à savoir

$$N^* \leqq C_2 X^{2\left(\frac{1}{k_{l+1}}+\ldots+\frac{1}{k_{n-1}}\right)-m+\varepsilon} \quad \ldots \ldots \quad (5)$$

Le § 2, dû à PISOT est entièrement consacré au problème de trouver des conditions suffisantes pour que l'inégalité (3) soit vérifiée. PISOT a trouvé deux résultats différents; nous parlerons d'abord de l'un de ces résultats.

Soit $s(k)$ un nombre naturel tel que le système

$$\left.\begin{array}{ll} \sum_{\sigma=1}^{s(k)} (f_\lambda(y_\sigma) - f_\lambda(z_\sigma)) = 0 & \\ |f_\lambda(y_\sigma)| \leqq X, \; |f_\lambda(z_\sigma)| \leqq X & (\sigma = 1, 2, \ldots, s(k)) \end{array}\right\}$$

possède au plus $C_3 X^{\frac{2}{k}s(k)-1+\varepsilon}$ solutions pour chaque λ $(1 \leqq \lambda \leqq l)$ tel que le degré de f_λ soit exactement k.

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[2]) Dans cette communication une solution veut toujours dire un système de nombres naturels vérifiant le système de relations en question.

Nous démontrerons que 2^{k-1} est une valeur possible pour $s(k)$. Mais comme on le sait il est possible de prendre pour $s(k)$ une valeur beaucoup plus petite, si k est assez grand.

Supposons que l'on puisse partager la matrice

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1l} \\ \cdot & \cdots & \cdot \\ b_{m1} & \cdots & b_{ml} \end{pmatrix}$$

en matrices partielles de rang m chacune, sans colonne commune deux à deux et supposons que dans chacune de ces matrices partielles les polynomes aient tous un même degré dit „associé" à la matrice partielle. Soit $\psi(k)$ le nombre des matrices partielles pour lesquelles le degré associé est égal à k.

Une condition suffisante pour que l'inégalité (3) soit vérifiée est

$$\sum_k \frac{\psi(k)}{s(k)} \geqq 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

où $\sum_k$ est étendu aux nombres k figurant comme degré dans au moins un polynome f_λ $(\lambda = 1, \ldots, l)$.

Cette condition a pour les petites valeurs de k le désavantage que le nombre l qui y figure doit être pris assez grand, à savoir $l \geqq 2^{k-1}\, m$ dans le cas particulier où tous les polynomes sont de même degré k; de sorte qu'alors le nombre n figurant dans (1) est $\geqq 2^k\, m + 1$. C'est pourquoi nous démontrerons aussi un autre théorème où l peut prendre une valeur un peu plus petite, à savoir $l \geqq (2^{k-1} - 1)m + 1$. Cette condition a par contre le désavantage que nous devons supposer que tous les polynomes soient de même degré k, de plus elle a l'inconvénient d'être plus difficile à énoncer que la précédente. Cette condition exige d'ailleurs aussi qu'un certain nombre de déterminants de la matrice B ne soient pas nuls.

Pour indiquer les déterminants dont il s'agit, nous allons définir certains ensembles $E_m(r)$ où r est l'une des valeurs $1, 2, \ldots, k$. Soit E un ensemble de systèmes d'entiers $(\nu_1, \nu_2, \ldots, \nu_a)$, E' un ensemble de systèmes d'entiers $(\nu'_1, \nu'_2, \ldots, \nu'_b)$. Nous désignerons par EE' l'ensemble de tous les systèmes de la forme $(\nu_1, \nu_2, \ldots, \nu_a, \nu'_1, \nu'_2, \ldots, \nu'_b)$, et par $E + E'$ l'ensemble des systèmes appartenant à au moins l'un des ensembles E ou E'.

$E_m(r)$ est un ensemble de systèmes $(\nu_1, \nu_2, \ldots, \nu_m)$ de m entiers $\nu_1, \nu_2, \ldots, \nu_m$ défini de la manière récurrente suivante:

Pour $1 \leqq r \leqq k$, $E_1(r)$ désignera tout ensemble formé par 2^{r-1} entiers différents.

Pour $m \geqq 2$, $r = 1$, $E_m(1)$ désignera tout ensemble de la forme $E_1(2)\, E_{m-1}(k-1)$ où aucun entier de $E_1(2)$ ne figure dans $E_{m-1}(k-1)$.

Pour $m \geqq 2$, $2 \leqq r \leqq k$, $E_m(r)$ désignera tout ensemble de la forme $E_1(r-1)E_{m-1}(k) + E_m(r-1)$ où aucun entier de $E_1(r-1)$ ne figure ni dans $E_{m-1}(k)$ ni dans $E_m(r-1)$.

Soit

$$l \geqq (2^{k-1}-1)m+1 \text{ si } k \geqq 3 \text{ et } l \geqq 2m \text{ si } k=2 \quad . \quad . \quad . \quad (7)$$

Supposons qu'il existe un ensemble $E_m(k)$ tel que pour chaque système $(\nu_1, \nu_2, \ldots, \nu_m)$ de cet ensemble les entiers $\nu_1, \nu_2, \ldots, \nu_m$ soient tous inférieurs ou égaux à l et que le déterminant

$$D(\nu_1, \ldots, \nu_m) = \begin{vmatrix} b_{1\nu_1} & b_{1\nu_2} & \ldots & b_{1\nu_m} \\ b_{2\nu_1} & b_{2\nu_2} & \ldots & b_{2\nu_m} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m\nu_1} & b_{m\nu_2} & \ldots & b_{m\nu_m} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

ne s'annule pas.

Le § 2 contient la démonstration que cette condition est suffisante pour que l'inégalité (3) soit vérifiée. Comme nous le verrons, à tout nombre l vérifiant (7) correspond au moins une matrice B pour laquelle cette condition est vérifiée.

Supposons maintenant que les inégalités (3) et (5) soient vérifiées. Dans ce cas nous pouvons appliquer le théorème fondamental A qui nous apprend que le facteur arithmétique $Z(t)$ peut être écrit comme un produit infini $\prod_p Q(p, t)$, étendu à tous les nombres premiers p, qui converge absolument et uniformément par rapport à $t_1, t_2, \ldots, t_m$, et qu'il existe deux nombres positifs fixes c_2 et ω tels que

$$|L(t) - b\,Z(t)\,A(t)| \leqq c_2\, X^{\frac{1}{k_1}+\ldots+\frac{1}{k_n}-m-\omega} \quad . \quad . \quad . \quad . \quad (9)$$

S'il est impossible de trouver n entiers $v_1, v_2, \ldots, v_n$ tels que

$$\sum_{\nu=1}^{n} b_{\mu\nu}\, v_\nu = t_\mu \qquad (\mu = 1, \ldots, m) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

il est clair que le système

$$\sum_{\nu=1}^{n} b_{\mu\nu}\, f_\nu(y_\nu) = t_\mu \qquad (\mu = 1, \ldots, m) \quad . \quad . \quad . \quad . \quad (11)$$

n'est pas résoluble. Ceci est aussi le cas s'il existe une puissance p^β d'un nombre premier p telle que le système

$$\sum_{\nu=1}^{n} b_{\mu\nu}\, f_\nu(y_\nu) \equiv t_\mu \;(mod.\; p^\beta) \qquad (\mu = 1, \ldots, m) \quad . \quad . \quad . \quad (12)$$

ne possède aucune solution.

Supposons donc dorénavant qu'il existe n entiers v_ν avec (10) et que le système de congruences (12) soit résoluble pour chaque puissance β d'un

nombre premier quelconque p. Sans nuire à la généralité nous pouvons supposer que le coefficient du terme de plus haut degré de $f_\nu(y_\nu)$ soit positif.

Distinguons maintenant deux cas.

I. Supposons que le système de relations

$$\sum_{\nu=1}^{n} b_{\mu\nu} u_\nu = 0 \quad (\mu = 1, \ldots, m) \qquad u_\nu > 0 \quad (\nu = 1, 2, \ldots, n) \quad . \quad . \quad (13)$$

possède $n-m$ solutions linéairement indépendantes. Dans ce cas on peut prendre les entiers fixes $t_1, t_2, \ldots, t_m$ arbitrairement, sous la condition que les systèmes (10) et (12) soient résolubles. Il suit du résultat général trouvé par VAN DER CORPUT dans l'article cité au renvoi [1]) que le facteur arithmétique est dans ce cas compris entre deux nombres positifs indépendants de X.

Le nombre $\Lambda(t)$ est de l'ordre de $X^{\frac{1}{k_1}+\ldots+\frac{1}{k_n}-m}$. Il résulte donc de l'inégalité (9) que $L(t)$ est aussi de l'ordre de $X^{\frac{1}{k_1}+\ldots+\frac{1}{k_n}-m}$ et croît indéfiniment avec X. Nous avons ainsi démontré le théorème suivant:

Théorème I: *Supposons vérifiées les inégalités (3) et (5) pour tout nombre positif fixe ε. Supposons que le système (13) possède $n-m$ solutions linéairement indépendantes.*

Chaque système t pour lequel (10) et (12) sont résolubles, p^β étant une puissance quelconque d'un nombre premier p arbitraire, peut être écrit d'une infinité de manières sous la forme (11) où $y_1, \ldots, y_n$ désignent des nombres naturels.

II. Le cas où on ne sait pas si le système (13) possède $n-m$ solutions linéairement indépendantes est plus difficile, puisqu'alors les nombres $t_1, \ldots, t_m$ ne peuvent peut-être pas être choisis fixes. Pour ce cas le résultat du § 3 est nécessaire. Dans ce § 3 nous nous bornons aux systèmes t tels que (12) possède pour chaque nombre premier p et pour tout nombre naturel β au moins une solution $y = (y_1, \ldots, y_n)$ à laquelle correspondent m nombres naturels différents $\nu_1, \ldots, \nu_m$ tous $\leqq n$, avec la propriété que le déterminant (8) ne s'annule pas et que chacune des m dérivées $f'_\nu(y_\nu)$ $(\nu = \nu_1, \ldots, \nu_m)$ soit divisible par ζ facteurs p au plus, où ζ ne dépend ni de X, ni de $t_1, \ldots$, ni de t_m. Nous supposons en outre qu'au système t peuvent être associés n entiers v_ν avec (10). Nous montrerons alors que le facteur arithmétique est compris entre deux nombres positifs indépendants de X et de $t_1, \ldots, t_m$.

Nous ne pouvons pas nous attendre dans ce cas à trouver toujours pour $\Lambda(t)$ l'ordre de $X^{\frac{1}{k_1}+\ldots+\frac{1}{k_n}-m}$. Au contraire, l'ordre de $\Lambda(t)$ dépend de l'ordre de grandeur des entiers $X, t_1, \ldots, t_m$. Nous pouvons cependant

choisir ces nombres de façon que $\Lambda(t)$ possède effectivement l'ordre indiqué. Supposons remplie l'inégalité moins exigeante

$$\Lambda(t) > C_4 X^{\frac{1}{k_1} + \dots + \frac{1}{k_n} - m - \varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

quel que soit le nombre positif fixe ε. Dans ce cas il résulte de (9) que $L(t)$ est du même ordre que $\Lambda(t)$, donc positif, et que l'on obtient le résultat suivant:

Théorème II: *Supposons vérifiées les inégalités (3), (5) et (14) pour tout nombre positif ε; supposons que (10) soit résoluble et que le système (12) possède pour chaque nombre premier p et pour tout nombre naturel β au moins une solution $y = (y_1, \dots, y_n)$ à laquelle correspondent m nombres naturels différents $\nu_1, \dots, \nu_m$, tous $\leqq n$, avec la propriété que le déterminant (8) ne s'annule pas et que chacune des m dérivées $f'_\nu(y_\nu)$ $(\nu = \nu_1, \dots, \nu_m)$ soit divisible par ζ facteurs p au plus, ζ étant indépendant de $X, t_1, \dots, t_m$.*

Si X est assez grand, le système t considéré peut être écrit sous la forme (11), où $y_1, \dots, y_n$ désignent des nombres naturels.

Remarque: Le théorème de WARING que tout nombre assez grand est la somme de n puissances $k^{\text{ièmes}}$, où n est un nombre naturel dépendant uniquement de k, est un cas très particulier du théorème II, si nous pouvons démontrer que la congruence

$$1 + \sum_{\nu = 1}^{n-1} y_\nu^k \equiv t \quad (mod.\, p^\beta)$$

est résoluble si n est assez grand. Cette démonstration se fait en quelques lignes [3].

§ 2. *Démonstration de l'inégalité (3).*

Dans ce paragraphe nous cherchons la borne supérieure (3) pour le nombre N de solutions du système (2).

Posons

$$S_\lambda = \sum_{y_\lambda} e^{2\pi i(\alpha_1 b_{1\lambda} + \alpha_2 b_{2\lambda} + \dots + \alpha_m b_{m\lambda}) f_\lambda(y_\lambda)}$$

où $\sum_{y_\lambda}$ est étendu aux entiers positifs y_λ tels que $|f_\lambda(y_\lambda)| \leqq X$; $\alpha_1, \alpha_2, \dots, \alpha_m$ sont des variables réelles et nous posons $d\alpha_1\, d\alpha_2 \dots d\alpha_m = d\alpha$.

Il est immédiat que

$$N = \int_0^1 \int_0^1 \dots \int_0^1 |S_1 S_2 \dots S_l|^2 \, d\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

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[3]) E. LANDAU, Vorlesungen über Zahlentheorie, T 1 (1927), p. 290.

Lemme 1: *Le nombre* $N^{\star\star}$ *de solutions* $(y_1, z_1, \ldots, y_{\lambda-1}, z_{\lambda-1}, y_{\lambda+1}, z_{\lambda+1}, \ldots, y_l, z_l)$ *du système obtenu en supprimant la colonne d'indice* λ *dans le système (2) vérifie l'inégalité* $N \leqq c_3 X^{\frac{2}{k_\lambda}} N^{\star\star}$, c_3 *étant un nombre indépendant de* X.

Le lemme 1 résulte de la formule (15) en remarquant qu'il existe une constante c_4 telle que $|S_\lambda| \leqq c_4 X^{\frac{1}{k_\lambda}}$.

Le lemme 1 montre que, si l'inégalité (3) est démontrée pour une certaine valeur de l, elle l'est pour toutes les valeurs de l plus grandes.

Démontrons d'abord que la condition (6), énoncée dans l'introduction, est suffisante pour que l'inégalité (3) soit vérifiée. Le lemme 1 nous permet de nous placer dans le cas où les matrices partielles de rang m de la matrice B ont toutes m colonnes. Désignons par Φ_h le produit des m quantités S_λ où λ parcourt les indices des colonnes de la $h^{\text{ième}}$ matrice carrée partielle et soit Ψ_k le produit des $\psi(k)$ quantités Φ_h où h correspond aux matrices partielles dont le degré associé est k. Alors

$$N = \int_0^1 \int_0^1 \ldots \int_0^1 \Pi_k |\Psi_k|^2 \, d\alpha,$$

où le produit Π_k est étendu aux valeurs de k associées à au moins une des matrices partielles.

En vertu des hypothèses il correspond à chacune de ces valeurs k un nombre $m(k) \geqq s(k)$ avec

$$\Sigma_k \frac{\psi(k)}{m(k)} = 1.$$

L'inégalité de SCHWARZ généralisée donne alors:

$$N \leqq \Pi_k \left(\int_0^1 \int_0^1 \ldots \int_0^1 |\Psi_k|^{\frac{2m(k)}{\psi(k)}} \, d\alpha \right)^{\frac{\psi(k)}{m(k)}} \quad \ldots \ldots \quad (16)$$

D'après la définition de Ψ_k on a

$$|\Psi_k| \leqq \Sigma_h |\Phi_h|^{\psi(k)} \quad \ldots \ldots \ldots \quad (17)$$

Ces formules montrent qu'il suffit de connaître une borne supérieure convenable du nombre

$$\int_0^1 \int_0^1 \ldots \int_0^1 |\Phi_h|^{2m(k)} \, d\alpha$$

pour toutes les valeurs de h qui entrent en considération. Comme

$m(k) \geqq s(k)$ nous pouvons nous contenter de chercher une borne supérieure convenable du nombre

$$N' = \int_0^1 \int_0^1 \dots \int_0^1 |\Phi_h|^{2s(k)} \, da.$$

Ce nombre N' est le nombre de solutions d'un système dont la matrice des coefficients

$$\begin{pmatrix} a_{11} \dots a_{1m} & a_{11} \dots a_{1m} & \dots\dots & a_{11} \dots a_{1m} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots & & \vdots \quad\quad \vdots \\ a_{m1} \dots a_{mm} & a_{m1} \dots a_{mm} & \dots\dots & a_{m1} \dots a_{mm} \end{pmatrix}$$

se compose de $s(k)$ matrices carrées de rang m toutes égales à la $h^{\text{ième}}$ matrice partielle de B. Le déterminant

$$\begin{vmatrix} a_{11} \dots a_{1m} \\ \dots\dots \\ a_{m1} \dots a_{mm} \end{vmatrix}$$

n'est pas nul par hypothèse, car c'est le déterminant de la $h^{\text{ième}}$ matrice carrée partielle de B. Il existe donc m^2 entiers $\zeta_{\varrho\mu}$ ($\varrho = 1, 2, \dots, m$; $\mu = 1, 2, \dots, m$), dont le déterminant n'est pas nul, tels que l'on ait

$$a'_{\varrho\tau} = \zeta_{\varrho 1} a_{1\tau} + \zeta_{\varrho 2} a_{2\tau} + \dots + \zeta_{\varrho m} a_{m\tau} \; (\tau = 1, 2, \dots, m)$$

avec

$$a'_{\varrho\tau} = a' \neq 0 \text{ si } \varrho = \tau \text{ et } a'_{\varrho\tau} = 0 \text{ si } \varrho \neq \tau.$$

Posons

$$a'_{\varrho\varkappa} = a'_{\varrho\tau} \text{ si } \varkappa \equiv \tau \, (mod.\ m).$$

Toute solution du système considéré (dont le nombre de solutions est N') est alors aussi solution du système suivant:

$$\left.\begin{array}{ll} \sum\limits_{\varkappa=1}^{m\,s(k)} a'_{\varrho\varkappa} (f_\lambda(y_\varkappa) - f_\lambda(z_\varkappa)) = 0 & (\varrho = 1, 2, \dots, m) \\ |f_\lambda(y_\varkappa)| \leqq X, \; |f_\lambda(z_\varkappa)| \leqq X & (\varkappa = 1, 2, \dots, m\,s(k)) \end{array}\right\} \quad . \quad . \quad (18)$$

λ étant l'indice de la colonne de la $h^{\text{ième}}$ matrice partielle pour laquelle on a $\lambda \equiv \varkappa$ (*mod* m).

Si N'' est le nombre de solutions du système (18) on a donc $N' \leqq N''$.

Or les valeurs des coefficients $a'_{\varrho\varkappa}$ montrent que le système (18) se décompose en m systèmes

$$\left.\begin{array}{ll} \sum\limits_{\sigma=1}^{s(k)} (f_\lambda(y_\sigma) - f_\lambda(z_\sigma)) = 0 & \\ |f_\lambda(y_\sigma)| \leqq X, \; |f_\lambda(z_\sigma)| \leqq X & (\sigma = 1, 2, \dots, s(k)) \end{array}\right\}$$

pour les m valeurs des indices λ des colonnes de la $h^{\text{ième}}$ matrice partielle de B.

En vertu de la définition de $s(k)$ on a alors:

$$N'' \leqq \left(C_5 X^{\frac{2}{k} s(k) - 1 + \frac{\varepsilon}{m}}\right)^m = C_6 X^{\frac{2}{k} m s(k) - m + \varepsilon}.$$

D'après $N' \leqq N''$ on a par conséquent

$$N' \leqq C_6 X^{\frac{2}{k} m s(k) - m + \varepsilon}.$$

Comme $m(k) \geqq s(k)$, on obtient

$$\int_0^1 \int_0^1 \dots \int_0^1 |\Phi_h|^{2m(k)} d\alpha \leqq C_7 X^{\frac{2}{k} m m(k) - m + \varepsilon}.$$

Enfin les formules (16) et (17) donnent

$$N \leqq \prod_k \left(\psi(k) C_7 X^{\frac{2}{k} m m(k) - m + \varepsilon}\right)^{\frac{\psi(k)}{m(k)}} \leqq C_1 \prod_k X^{\frac{2\psi(k)}{k} m - \frac{\psi(k)}{m(k)}(m - \varepsilon)}.$$

En tenant compte de la définition de $m(k)$ et de ce que

$$\sum_k \frac{m \psi(k)}{k} = \frac{1}{k_1} + \dots + \frac{1}{k_l}$$

il vient:

$$N \leqq C_1 X^{2\left(\frac{1}{k_1} + \dots + \frac{1}{k_l}\right) - m + \varepsilon}$$

et nous avons démontré que l'inégalité (6) est suffisante pour que l'inégalité (3) soit vérifiée.

Mathematics. — *Zur Theorie der hypergeometrischen Funktionen.* Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 25, 1939.)

§ 1. In der von $z=1$ bis $z=\infty$ längs der reellen Achse aufgeschnittenen z-Ebene gelten bekanntlich für die hypergeometrische Funktion ${}_2F_1(a, b; c; z)$ die Integraldarstellungen

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-a)\,\Gamma(a)} \int_0^1 (1-zu)^{-b}\,(1-u)^{c-a-1}\,u^{a-1}\,du \\ [\Re(c) > \Re(a) > 0] \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

und

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-b)\,\Gamma(b)} \int_0^1 (1-zu)^{-a}\,(1-u)^{c-b-1}\,u^{b-1}\,du \\ [\Re(c) > \Re(b) > 0]. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Im Gegensatz zu den linken Seiten bekommen die rechten Seiten bei Verwechselung von a und b eine andere Gestalt. ERDÉLYI [1]) aber hat neuerdings die folgende, beiderseits symmetrische Erweiterung von (1) und (2) abgeleitet

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-\sigma)\,\Gamma(\sigma)} \int_0^1 {}_2F_1(a, b; \sigma; zu)\,(1-u)^{c-\sigma-1}\,u^{\sigma-1}\,du \\ [\Re(c) > \Re(\sigma) > 0]. \end{aligned}\right\} \quad . \quad (3)$$

Es ist leicht einzusehen, dass (1) und (2) Spezialfälle von (3) sind; denn wegen

$${}_2F_1(\lambda, \mu; \lambda; w) = (1-w)^{-\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

geht (3) für $\sigma = a$ in (1), für $\sigma = b$ aber in (2) über.

Ich habe eine andere Erweiterung von (1) und (2) konstruiert, nämlich

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(\alpha)\,\Gamma(c)}{\Gamma(\alpha+c-a-b)\,\Gamma(a)\,\Gamma(b)} \int_0^1 (1-zu)^{-\alpha} \\ \times {}_2F_1(\alpha-b, c-b; \alpha+c-a-b; 1-u)\,(1-u)^{\alpha+c-a-b-1}\,u^{a-1}\,du. \end{aligned}\right\} \quad . \quad . \quad (5)$$

[1]) ERDÉLYI, [5], 203.

Hierin sind $\Re(a)$ und $\Re(b) > 0$; α ist beliebig mit

$$\Re(\alpha + c - a - b) > 0, \quad \alpha \neq 0, -1, -2, \ldots .$$

Die rechte Seite von (5) ist wegen

$${}_2F_1(\lambda, \mu; \nu; 1-u) = u^{\nu-\lambda-\mu}\, {}_2F_1(\nu-\mu, \nu-\lambda; \nu; 1-u) \quad . \quad . \quad . \quad (6)$$

symmetrisch in Bezug auf a und b. Setzt man $\alpha = b$, so findet man (1); für $\alpha = a$ geht (5) infolge (4) in (2) über [2]).

Eine Verallgemeinerung von (5) ist [3])

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta-a-b)\,\Gamma(a)\,\Gamma(b)} \int_0^1 {}_2F_1(\alpha, \beta; c; zu) \\ \times\, {}_2F_1(\alpha-b, \beta-b; \alpha+\beta-a-b; 1-u)\,(1-u)^{\alpha+\beta-a-b-1}\, u^{a-1}\, du. \end{aligned}\right\} \quad . \quad . \quad (7)$$

In dieser Relation ist $\Re(a) > 0$ und $\Re(b) > 0$; α und β sind beliebige Zahlen mit

$$\Re(\alpha + \beta - a - b) > 0, \quad \alpha \neq 0, -1, -2, \ldots, \beta \neq 0, -1, -2, \ldots .$$

Der Spezialfall mit $\beta = c$ von (7) ergibt (5); nimmt man $\beta = b$, so erhält man

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-a)\,\Gamma(a)} \int_0^1 {}_2F_1(\alpha, b; c; zu)\,(1-u)^{\alpha-a-1}\, u^{a-1}\, du \\ [\Re(\alpha) > \Re(a) > 0]. \end{aligned}\right\} \quad . \quad (8)$$

Diese Beziehung, die für $\alpha = c$ in (1) übergeht, kommt auch bei ERDÉLYI vor [4]).

Eine sehr allgemeine Formel, die (3) und (7) als Sonderfälle enthält, ist

$$\left.\begin{aligned} {}_2F_1(a, b; c; z) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta-\sigma-\tau)\,\Gamma(\sigma)\,\Gamma(\tau)} \int_0^1 {}_4F_3(\alpha, \beta, a, b; \sigma, \tau, c; zu) \\ \times\, {}_2F_1(\alpha-\tau, \beta-\tau; \alpha+\beta-\sigma-\tau; 1-u)\,(1-u)^{\alpha+\beta-\sigma-\tau-1}\, u^{\sigma-1}\, du. \end{aligned}\right\} \quad . \quad (9)$$

Hierin sind α, β, σ und τ beliebig mit

$$\Re(\alpha + \beta - \sigma - \tau) > 0, \; \Re(\sigma) > 0, \; \Re(\tau) > 0, \quad . \quad . \quad . \quad . \quad (10)$$

$$\alpha \neq 0, -1, -2, \ldots, \; \beta \neq 0, -1, -2, \ldots \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Nimmt man nun $\sigma = a$ und $\tau = b$, so findet man (7); ein verwandtes

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[2]) Formel (1) folgt aus (5), falls $\Re(a)$, $\Re(b)$ und $\Re(c-a) > 0$ sind; die Bedingung $\Re(b) > 0$ kann aber sofort mit Hilfe von analytischen Fortsetzung beseitigt werden. Bei (2) geht es analog.

[3]) Die rechte Seite von (7) ist wegen (6) eine symmetrische Funktion von a und b.

[4]) ERDÉLYI, [6], 270, Formel (2.6).

Resultat kommt zum Vorschein, wenn man $\beta = c$ und $\tau = a$ setzt. Der Spezialfall mit $\beta = \tau$ von (9) liefert

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(a)}{\Gamma(a-\sigma)\,\Gamma(\sigma)} \int_0^1 {}_3F_2(a, a, b; \sigma, c; zu)(1-u)^{a-\sigma-1}\, u^{\sigma-1}\, du \; ^{5)}. \quad (12) $$

Diese Beziehung, gültig für beliebige Werte von a und σ mit $\Re(a) > \Re(\sigma) > 0$, geht für $a = c$ in (3) und für $\sigma = a$ in (8) über.

Der Beweis von (9) geht wie folgt: Ist $\Re(\nu - \lambda - \mu) > 0$, so kann die hypergeometrische Funktion ${}_2F_1(\lambda, \mu; \nu; x)$ in eine für $0 \leqq x \leqq 1$ gleichmässig konvergente Potenzreihe entwickelt werden. Ist überdies $\Re(\nu) > 0$ und $\Re(\varrho) > 0$, so darf man daher im Integral

$$ \int_0^1 {}_2F_1(\lambda, \mu; \nu; x)\, x^{\nu-1}(1-x)^{\varrho-1}\, dx $$

die Funktion ${}_2F_1$ entwicklen und gliedweise integrieren; man findet dann den Wert

$$ \frac{\Gamma(\nu)\,\Gamma(\varrho)}{\Gamma(\nu+\varrho)}\, {}_2F_1(\lambda, \mu; \nu+\varrho; 1) = \frac{\Gamma(\nu)\,\Gamma(\varrho)\,\Gamma(\nu+\varrho-\lambda-\mu)}{\Gamma(\nu+\varrho-\mu)\,\Gamma(\nu+\varrho-\lambda)}. $$

Unter der Annahme [6])

$$ \Re(\nu) > 0, \; \Re(\varrho) > 0, \; \Re(\nu+\varrho-\lambda-\mu) > 0 \; . \; . \; . \; . \quad (13) $$

erhält man also

$$ \int_0^1 {}_2F_1(\lambda, \mu; \nu; x)\, x^{\nu-1}(1-x)^{\varrho-1}\, dx = \frac{\Gamma(\nu)\,\Gamma(\varrho)\,\Gamma(\nu+\varrho-\lambda-\mu)}{\Gamma(\nu+\varrho-\mu)\,\Gamma(\nu+\varrho-\lambda)}. \quad (14) $$

Genügen a, β, σ und τ den Bedingungen (10) und ist $l \geqq 0$, so gilt daher

$$ \int_0^1 {}_2F_1(a-\tau, \beta-\tau; a+\beta-\sigma-\tau; 1-u)\,(1-u)^{a+\beta-\sigma-\tau-1}\, u^{\sigma+l-1}\, du $$

$$ = \frac{\Gamma(a+\beta-\sigma-\tau)\,\Gamma(\sigma+l)\,\Gamma(\tau+l)}{\Gamma(a+l)\,\Gamma(\beta+l)}. $$

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5) Man vergl. auch Formel (7) der Arbeit [16].

6) Das Integral auf der linken Seite von (14) konvergiert, falls die Voraussetzungen (13) erfüllt sind; denn für $x \to 1$ hat man

$$ {}_2F_1(\lambda, \mu; \nu; x) = O(1) + O(|1-x|^{\Re(\nu-\lambda-\mu)}). $$

Der Beweis von (9) erfolgt nun für $|z| < 1$ durch Entwicklung der im Integranden auftretenden Funktion ${}_4F_3(zu)$ und gliedweise Integration; aus dem Prinzip der analytischen Fortsetzung ergibt sich dann, dass (9) gültig ist in der von $z = 1$ bis $z = \infty$ aufgeschnittenen z-Ebene.

Auf analoge Weise wie (9) findet man das allgemeinere Resultat:
Genügen α, β, σ und τ den Bedingungen (10) und (11), so ist

$$\left.\begin{aligned} {}_pF_q\binom{a_1,\ldots,a_p;}{c_1,\ldots,c_q;z} &= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta-\sigma-\tau)\,\Gamma(\sigma)\,\Gamma(\tau)}\int_0^1 {}_{p+2}F_{q+2}\binom{\alpha,\beta,a_1,\ldots,a_p;}{\sigma,\tau,c_1,\ldots,c_q;zu} \\ &\times {}_2F_1(\alpha-\tau,\beta-\tau;\alpha+\beta-\sigma-\tau;1-u)\,(1-u)^{\alpha+\beta-\sigma-\tau-1}\,u^{\tau-1}\,du. \end{aligned}\right\}\quad(15)$$

Ebenso beweist man:
Sind die Voraussetzungen (10) erfüllt, so ist [7]

$$\left.\begin{aligned} &\frac{\Gamma(\alpha+\beta-\sigma-\tau)\,\Gamma(\sigma)\,\Gamma(\tau)}{\Gamma(\alpha)\,\Gamma(\beta)}\,{}_{p+2}F_{q+2}\binom{a_1,\ldots,a_p,\sigma,\tau;}{c_1,\ldots,c_q,\alpha,\beta;z} \\ &= \int_0^1 {}_pF_q\binom{a_1,\ldots,a_p;}{c_1,\ldots,c_q;zu}\cdot{}_2F_1\binom{\alpha-\tau,\beta-\tau;}{\alpha+\beta-\sigma-\tau;1-u}(1-u)^{\alpha+\beta-\sigma-\tau-1}\,u^{\tau-1}\,du. \end{aligned}\right\}\quad(16)$$

Ist $p = q+1$, so gelten (15) und (16) in der von $z = 1$ bis $z = \infty$ aufgeschnittenen z-Ebene; für $p \leqq q$ ist z beliebig.

Formel (16) kommt auch zum Vorschein, wenn man in (15) p durch $p+2$ und q durch $q+2$ ersetzt und überdies $a_{p+1} = \sigma$, $a_{p+2} = \tau$, $c_{q+1} = \alpha$ und $c_{q+2} = \beta$ setzt. Man findet (15), wenn man in (16) p durch $p+2$ und q durch $q+2$ ersetzt und ausserdem $a_{p+1} = \alpha$, $a_{p+2} = \beta$, $c_{q+1} = \sigma$ und $c_{q+2} = \tau$ nimmt.

§ 2. Ich gebe jetzt einige Anwendungen von (16).

1. *Integraldarstellungen für* $M_{k,m}(z)\,M_{-k,m}(z)$. Das Produkt der WHITTAKERschen Funktionen $M_{k,m}(z)$ und $M_{-k,m}(z)$ ist bekanntlich gleich [8]

$$M_{k,m}(z)\,M_{-k,m}(z) = z^{1+2m}\,{}_2F_3(\tfrac{1}{2}+m+k,\tfrac{1}{2}+m-k;1+2m,\tfrac{1}{2}+m,1+m;\tfrac{1}{4}z^2).$$

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[7]) Die linke Seite von (16) hat auch einen Sinn, falls α oder $\beta = 0, -1, -2, \ldots\ldots$ ist. Es gilt z.B.

$$\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\,{}_2F_3\binom{\sigma,\tau;}{c,\alpha,\beta;z} = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} + \frac{\sigma\tau z}{c\,\Gamma(\alpha+1)\,\Gamma(\beta+1)} + \frac{\sigma(\sigma+1)\,\tau(\tau+1)\,z^2}{2!\,c(c+1)\,\Gamma(\alpha+2)\,\Gamma(\beta+2)} + \ldots.$$

[8]) MEIJER, [13], Formeln (9) und (2).

Hieraus und aus (16) geht hervor, falls $\Re(\frac{1}{2}+m\pm k)>0$ ist,

$$\left.\begin{aligned} M_{k,m}(z)M_{-k,m}(z) = {} & \frac{z^{1+2m}\,\Gamma(\frac{1}{2}+m)\,\Gamma(1+m)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)}\int\limits_0^1 {}_0F_1(1+2m;\tfrac{1}{4}z^2u) \\ & \times {}_2F_1(k,\tfrac{1}{2}+k;\tfrac{1}{2};1-u)\,(1-u)^{-\frac{1}{2}}\,u^{m+k-\frac{1}{2}}\,du \end{aligned}\right\} \quad (17)$$

und

$$\left.\begin{aligned} M_{k,m}(z)\,M_{-k,m}(z) = {} & \frac{z^{1+2m}\,\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)}\int\limits_0^1 {}_0F_1(\tfrac{1}{2}+m;\tfrac{1}{4}z^2u) \\ & \times {}_2F_1(\tfrac{1}{2}+k,\tfrac{1}{2}+m+k;1+m;1-u)\,(1-u)^m\,u^{m+k-\frac{1}{2}}\,du. \end{aligned}\right\} \quad (18)$$

Nun ist [9])

$${}_0F_1(1+\nu;\tfrac{1}{4}\zeta^2) = 2^\nu\,\zeta^{-\nu}\,\Gamma(1+\nu)\,I_\nu(\zeta),$$

wo $I_\nu(\zeta)$ die BESSELsche Funktion imaginären Argumentes bezeichnet.

Ferner hat man [10])

$${}_2F_1(k,\tfrac{1}{2}+k;\tfrac{1}{2};\operatorname{tgh}^2 t)\operatorname{sech}^{2k} t = \cosh 2kt$$

und [11])

$${}_2F_1(\tfrac{1}{2}+k,\tfrac{1}{2}+m+k;1+m;\operatorname{tgh}^2 t) = \Gamma(1+m)\operatorname{tgh}^{-m} t\cosh^{2k+1} t\,P_{k-\frac{1}{2}}^{-m}(\cosh 2t);$$

hierin bedeutet $P_\nu^\mu(\zeta)$ die zugeordnete LEGENDREsche Funktion erster Art.

Setzt man nun $u=\operatorname{sech}^2 t$ in (17) und (18), so findet man also [12])

$$M_{k,m}(z)M_{-k,m}(z) = \frac{2z\,\Gamma^2(1+2m)}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)}\int\limits_0^\infty I_{2m}(z\operatorname{sech} t)\cosh 2kt\operatorname{sech} t\,dt \quad (19)$$

und

$$\left.\begin{aligned} M_{k,m}(z)\,M_{-k,m}(z) = {} & \frac{2^{-m+\frac{1}{2}}\,z^{m+1/2}\,\Gamma^2(1+2m)\sqrt{\pi}}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)} \\ \times \int\limits_0^\infty I_{m-\frac{1}{2}}(z\operatorname{sech} t)\,& P_{k-\frac{1}{2}}^{-m}(\cosh 2t)\sinh^{m+1} t\cosh^{-2m-3/2} t\,dt. \end{aligned}\right\}. \quad (20)$$

Formel (19) war schon bekannt [13]).

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[9]) WATSON, [18], 77, Formel (2).

[10]) GAUSS, [7], 127, Formel (XXII).

[11]) HOBSON, [8], 210, Formel (41).

[12]) Ich benutze die Beziehung $\Gamma(\frac{1}{2}+m)\Gamma(1+m) = 2^{-2m}\sqrt{\pi}\,\Gamma(1+2m)$.

[13]) MEIJER, [15].

2. *Integraldarstellungen für die Funktion* $s_{\mu,\nu}(z)$. Ist $\mu \pm \nu \neq -1, -3, -5, \ldots$, so wird die LOMMELsche Funktion $s_{\mu,\nu}(z)$ definiert durch [14])

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(1+\mu+\nu)(1+\mu-\nu)}\, {}_1F_2(1; \tfrac{3}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, \tfrac{3}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu; -\tfrac{1}{4}z^2).$$

Setzt man nun zur Abkürzung

$$r_{\mu,\nu}(z) = \frac{s_{\mu,\nu}(z)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)}, \quad . \quad . \quad . \quad (21)$$

so gilt also

$$r_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4\,\Gamma(\frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)}\, {}_1F_2(1; \tfrac{3}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, \tfrac{3}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu; -\tfrac{1}{4}z^2) \quad (22)$$

und diese Funktion hat auch einen Sinn für $\mu \pm \nu = -1, -3, -5, \ldots$ [15]).

Nun folgt aus (16), mit $p=0$, $q=1$ und $c_1=\sigma$ angewendet, falls α, β, σ und τ den Bedingungen (10) genügen,

$$\left.\begin{aligned} &\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\, {}_1F_2(\tau; \alpha, \beta; w) = \frac{1}{\Gamma(\alpha+\beta-\sigma-\tau)\,\Gamma(\sigma)\,\Gamma(\tau)} \\ &\times \int_0^1 {}_0F_1(\sigma; wu)\,.\, {}_2F_1(\alpha-\tau, \beta-\tau; \alpha+\beta-\sigma-\tau; 1-u)(1-u)^{\alpha+\beta-\sigma-\tau-1}\, u^{\sigma-1}\, du. \end{aligned}\right\} (23)$$

Ich brauche ferner die bekannten Relationen

$${}_0F_1(\tfrac{1}{2}; -\tfrac{1}{4}\zeta^2) = \cos\zeta, \qquad {}_0F_1(\tfrac{3}{2}; -\tfrac{1}{4}\zeta^2) = \zeta^{-1}\sin\zeta \quad . \quad . \quad (24)$$

und

$${}_0F_1(1+\nu; -\tfrac{1}{4}\zeta^2) = 2^\nu \zeta^{-\nu}\, \Gamma(1+\nu)\, J_\nu(\zeta). \quad . \quad . \quad . \quad . \quad (25)$$

Schliesslich benutze ich noch die für $0 < \varphi < \frac{1}{2}\pi$ gültige Beziehungen [16])

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14) WATSON, [18], 346.

15) Man vergl. Fussnote 7).

16) Für $-1 < x < 1$ wird die LEGENDREsche Funktion $P_n^m(x)$ gewöhnlich definiert durch (HOBSON, [8], 227)

$$P_n^m(x) = \frac{1}{\Gamma(1-m)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} {}_2F_1(1+n, -n; 1-m; \tfrac{1}{2}-\tfrac{1}{2}x).$$

Setzt man hierin $x = \cos 2\varphi$ und benutzt man (6), so findet man (28). Relation (26) folgt mit Hilfe von (KUMMER, [9], 78, Formel (57))

$${}_2F_1(2a, 2b; \tfrac{1}{2}+a+b; t) = {}_2F_1\{a, b; \tfrac{1}{2}+a+b; 4t(1-t)\}$$

aus (28) (mit $\frac{1}{2}\varphi$ statt φ angewendet). Die Anwendung von (6) auf (26) liefert (27).

$$ {}_2F_1\left(\tfrac{1}{2}-\tfrac{1}{2}m+\tfrac{1}{2}n, -\tfrac{1}{2}m-\tfrac{1}{2}n; 1-m; \sin^2\varphi\right) = 2^{-m}\,\Gamma(1-m)\sin^m\varphi\, P_n^m(\cos\varphi), \quad . \quad . \tag{26} $$

$$ {}_2F_1\left(1-\tfrac{1}{2}m+\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}m-\tfrac{1}{2}n; 1-m; \sin^2\varphi\right) = 2^{-m}\,\Gamma(1-m)\sin^m\varphi\cos^{-1}\varphi\, P_n^m(\cos\varphi) \tag{27} $$

und

$$ {}_2F_1(1-m+n, -m-n; 1-m; \sin^2\varphi) = \Gamma(1-m)\sin^m\varphi\cos^m\varphi\, P_n^m(\cos 2\varphi). \quad . \quad . \tag{28} $$

Ich wende nun (23) (mit $\tau = 1$, $\alpha = \tfrac{3}{2} + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu$, $\beta = \tfrac{3}{2} + \tfrac{1}{2}\mu - \tfrac{1}{2}\nu$, $w = -\tfrac{1}{4}z^2$ und $u = \cos^2\varphi$) fünfmal an, nämlich mit $\sigma = \tfrac{1}{2}$, $\sigma = \tfrac{3}{2}$, $\sigma = 1 + \tfrac{1}{2}\mu$, $\sigma = \tfrac{3}{2} + \tfrac{1}{2}\mu$ und $\sigma = \tfrac{1}{2} + \tfrac{1}{2}\mu$. Mit Rücksicht auf (22), (24), (25), (26), (27) und (28) finde ich dann [17])

$$ r_{\mu,\nu}(z) = \frac{2^{\mu-\frac{1}{2}}\, z^{\mu+1}}{\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} \cos(z\cos\varphi)\, P_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}(\cos\varphi)\sin^{\mu+3/2}\varphi\, d\varphi, \tag{29} $$

$$ r_{\mu,\nu}(z) = \frac{2^{\mu-\frac{1}{2}}\, z^{\mu}}{\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} \sin(z\cos\varphi)\, P_{\nu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(\cos\varphi)\sin^{\mu+\frac{1}{2}}\varphi\, d\varphi, \quad . \tag{30} $$

$$ r_{\mu,\nu}(z) = 2^{\frac{1}{2}\mu-1}\, z^{\frac{1}{2}\mu+1} \int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}\mu}(z\cos\varphi)\, P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\frac{1}{2}\mu}(\cos 2\varphi)\sin^{\frac{1}{2}\mu+1}\varphi\cos\varphi\, d\varphi, \tag{31} $$

$$ \left.\begin{aligned} r_{\mu,\nu}(z) &= 2^{\frac{1}{2}\mu-3/2}\, z^{\frac{1}{2}\mu+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}\mu+\frac{1}{2}}(z\cos\varphi) \\ &\times \left\{ P_{\frac{1}{2}\nu}^{\frac{1}{2}-\frac{1}{2}\mu}(\cos 2\varphi) + P_{-\frac{1}{2}\nu}^{\frac{1}{2}-\frac{1}{2}\mu}(\cos 2\varphi) \right\} \sin^{\frac{1}{2}\mu+\frac{1}{2}}\varphi\, d\varphi \quad {}^{18)}, \end{aligned}\right\} \quad . \tag{32} $$

$$ \left.\begin{aligned} \nu\, r_{\mu,\nu}(z) &= 2^{\frac{1}{2}\mu-5/2}\, z^{\frac{1}{2}\mu+3/2} \int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}\mu-\frac{1}{2}}(z\cos\varphi) \\ &\times \left\{ (1+\nu+\mu) P_{\frac{1}{2}\nu}^{-\frac{1}{2}\mu-\frac{1}{2}}(\cos 2\varphi) - (1-\nu+\mu) P_{-\frac{1}{2}\nu}^{-\frac{1}{2}\mu-\frac{1}{2}}(\cos 2\varphi) \right\} \sin^{\frac{1}{2}\mu+3/2}\varphi\, d\varphi. \end{aligned}\right\} \tag{33} $$

Diese Integraldarstellungen scheinen noch nicht angegeben worden zu sein; entsprechende Beziehungen für die LOMMELsche Funktion $S_{\mu,\nu}(z)$ waren aber schon bekannt [19]). Die Spezialfälle mit $\nu - \mu = 2, 4, 6, \ldots$ von (29) kommen mit ganz anderen Bezeichnungen bei SZYMAŃSKI vor [20]). Formel (30) mit $\mu = \nu$ war auch bekannt [21]).

[17]) Die Voraussetzungen, worunter diese Beziehungen gelten, sind: $\Re(\mu) > -\tfrac{3}{2}$ in (29), $\Re(\mu) > -\tfrac{1}{2}$ in (30), $\Re(\mu) > -2$ in (31) und $\Re(\mu) > -1$ in (32) und in (33).

[18]) Bei der Ableitung von (32) und (33) benutze ich noch die Formeln (13) bezw. (14) meiner Arbeit **[14]**.

[19]) MEIJER, **[14]**.

[20]) SZYMAŃSKI, **[17]**, 76, Formel (4).

[21]) WATSON, **[18]**, 328, Formel (1); SZYMAŃSKI, **[17]**, 82, Formel (4).

Nun gilt bekanntlich [22])

$$ {}_2F_1(\lambda, -\lambda; \tfrac{1}{2}; \sin^2\varphi) = \cos 2\lambda\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

und

$$ {}_2F_1(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\lambda; \tfrac{3}{2}; \sin^2\varphi) = \frac{\sin 2\lambda\varphi}{2\lambda\sin\varphi}.$$

Hieraus und aus (26) geht hervor

$$P_n^{\frac{1}{2}}(\cos\varphi) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(n+\frac{1}{2})\varphi}{\sin^{\frac{1}{2}}\varphi}$$

und

$$(n+\tfrac{1}{2})\,P_n^{-\frac{1}{2}}(\cos\varphi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(n+\frac{1}{2})\varphi}{\sin^{\frac{1}{2}}\varphi}.$$

Aus (29) mit $\mu = -1$ bezw. $\mu = 0$ ergibt sich somit

$$r_{-1,\nu}(z) = \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi} \cos(z\cos\varphi)\cos\nu\varphi\, d\varphi \quad . \quad . \quad . \quad . \quad . \quad (35)$$

und

$$\nu\, r_{0,\nu}(z) = \frac{z}{\pi}\int_0^{\frac{1}{2}\pi} \cos(z\cos\varphi)\sin\nu\varphi\sin\varphi\, d\varphi. \quad . \quad . \quad . \quad (36)$$

Auf analoge Weise folgt aus (30) mit $\mu = 0$ bezw. $\mu = 1$

$$r_{0,\nu}(z) = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\cos\varphi)\cos\nu\varphi\, d\varphi \quad . \quad . \quad . \quad . \quad . \quad (37)$$

und

$$\nu\, r_{1,\nu}(z) = \frac{2z}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\cos\varphi)\sin\nu\varphi\sin\varphi\, d\varphi. \quad . \quad . \quad . \quad (38)$$

Die Beziehungen (35) und (38) kommen auch zum Vorschein, wenn man $\mu = -1$ bezw. $\mu = 1$ setzt in (31) [23]). Nimmt man $\mu = 0$ in (32) und in (33), so findet man (37) bezw. (36).

(35) und (37) waren schon bekannt [24]).

[22]) GAUSS, [**7**], 127, Formeln (XX) und (XVI); siehe auch MACROBERT, [**10**], 633.

[23]) $J_{-\frac{1}{2}}(\zeta) = \left(\dfrac{2}{\pi\zeta}\right)^{\frac{1}{2}}\cos\zeta, \qquad J_{\frac{1}{2}}(\zeta) = \left(\dfrac{2}{\pi\zeta}\right)^{\frac{1}{2}}\sin\zeta.$

[24]) WATSON, [**18**], 310, Formeln (15) und (16).

3. *Integraldarstellungen für $J_\mu(z)\,J_\nu(z)$.* Dieses Produkt ist gleich [25])

$$J_\mu(z)\,J_\nu(z)=\frac{2^{-\mu-\nu}\,z^{\mu+\nu}}{\Gamma(1+\mu)\,\Gamma(1+\nu)}\,{}_2F_3\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu,\ 1+\frac{1}{2}\mu+\frac{1}{2}\nu;\\ 1+\mu,\ 1+\nu,\ 1+\mu+\nu;\ -z^2\end{matrix}\right). \qquad (39)$$

Hieraus und aus (16) folgt, falls $\Re(\mu+\nu)>-1$ ist,

$$J_\mu(z)\,J_\nu(z)=\frac{2^{-\mu-\nu}\,z^{\mu+\nu}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu)}\int_0^1 {}_0F_1(1+\mu+\nu;-z^2u)$$

$$\times\,{}_2F_1(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu,\tfrac{1}{2}\nu-\tfrac{1}{2}\mu;\tfrac{1}{2};1-u)\,(1-u)^{-\frac{1}{2}}\,u^{\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}}\,du.$$

Setzt man nun noch $u=\cos^2\varphi$, so erhält man mit Rücksicht auf (25) und (34)

$$J_\mu(z)\,J_\nu(z)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}J_{\mu+\nu}(2z\cos\varphi)\cos(\mu-\nu)\varphi\,d\varphi;$$

diese Beziehung ist schon lange bekannt [26]).

Auf analoge Weise findet man mit Hilfe von (25) und (26), falls $\Re(\mu)>-\frac{1}{2}$ und $\Re(\mu+\nu)>-1$ ist,

$$J_\mu(z)\,J_\nu(z)=\frac{2^{\mu+\frac{1}{2}}\,z^\mu}{\sqrt{\pi}}\int_0^{\frac{1}{2}\pi}J_\nu(2z\cos\varphi)\,P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\cos\varphi)\sin^{\mu+\frac{1}{2}}\varphi\cos^\mu\varphi\,d\varphi.$$

Diese Integraldarstellung ist meines Wissens noch nicht gegeben worden; entsprechende Formeln für die Produkte $K_\mu(z)\,K_\nu(z)$, $H^{(1)}_\mu(z)\,H^{(2)}_\nu(z)$, $H^{(2)}_\mu(z)\,H^{(1)}_\nu(z)$ und $K_\mu(z)\,I_\nu(z)$ waren aber schon bekannt [27]).

Nun hat man wegen (16), mit $p=1$, $q=2$, $\alpha=a_1=1$, $\beta=b$, $c_1=c$ und $c_2=d$ angewendet, falls $\Re(b-\sigma-\tau)>-1$, $\Re(\sigma)>0$ und $\Re(\tau)>0$ ist,

$$\frac{1}{\Gamma(b)}\,{}_2F_3\left(\begin{matrix}\sigma,\tau;\\ b,c,d;z\end{matrix}\right)=\frac{1}{\Gamma(1+b-\sigma-\tau)\,\Gamma(\sigma)\,\Gamma(\tau)}$$

$$\times\int_0^1 {}_1F_2(1;c,d;zu)\,.\,{}_2F_1(1-\tau,b-\tau;1+b-\sigma-\tau;1-u)\,(1-u)^{b-\sigma-\tau}\,u^{\sigma-1}\,du.$$

Aus dieser Beziehung (mit $u=\cos^2\varphi$) und (39) folgt mittels (22) und (34), falls $\Re(\mu+\nu)>-1$ ist,

$$J_\mu(z)\,J_\nu(z)=\frac{2^{3-\mu-\nu}}{\pi}\int_0^{\frac{1}{2}\pi}r_{\mu+\nu-1,\mu-\nu}(2z\cos\varphi)\cos(\mu+\nu)\varphi\,d\varphi,\qquad (40)$$

wo $r_{\mu,\nu}(z)$ die durch (21) definierte Funktion bezeichnet.

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25) WATSON, [18], 147.

26) WATSON, [18], 150.

27) MEIJER, [11], §§ 4 und 5; [12], Formel (63).

In ähnlicher Weise findet man mit Hilfe von (22) und (26), falls $\Re(\mu) < \frac{1}{2}$ und $\Re(\mu+\nu) > -1$ ist,

$$J_\mu(z) J_\nu(z) = \frac{2^{-3\mu-\nu+5/2} z^{-\mu}}{\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} r_{2\mu+\nu-1,\nu}(2z\cos\varphi)\, P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\varphi) \sin^{\frac{1}{2}-\mu}\varphi \cos^{-\mu}\varphi\, d\varphi. \quad (41)$$

Die Relationen (40) und (41) scheinen neu zu sein.

4. *Die* GEGENBAUER*sche Integraldarstellung der* BESSEL*schen Funktion.* Diese Integraldarstellung lautet wie folgt [28])

$$J_{\nu+n}(z) = \frac{(-i)^n\, n!\, (\frac{1}{2}z)^\nu}{\sqrt{\pi}\, \Gamma(\frac{1}{2}+\nu) \prod\limits_{h=0}^{n-1} (2\nu+h)} \int_{-1}^{1} e^{izt}\, C_n^\nu(t)\, (1-t^2)^{\nu-\frac{1}{2}}\, dt; \quad (42)$$

hierin ist $\Re(\nu) > -\frac{1}{2}$, $n = 0, 1, 2, \ldots$ [29]) und $C_n^\nu(t)$ bezeichnet den Koeffizienten von x^n in der Entwicklung von $(1-2tx+x^2)^{-\nu}$ nach steigenden Potenzen von x.

Ich werde zeigen, dass (42) ein Spezialfall von (16) ist. Man hat nämlich [30])

$$\frac{(-1)^n\, n!}{\prod\limits_{h=0}^{n-1} (2\nu+h)}\, C_n^\nu(t) = {}_2F_1(-n, 2\nu+n; \tfrac{1}{2}+\nu; \tfrac{1}{2}+\tfrac{1}{2}t).$$

Die rechte Seite von (42) ist also gleich (ich setze $t = 1-2u$)

$$\frac{i^n (2z)^\nu e^{iz}}{\sqrt{\pi}\, \Gamma(\frac{1}{2}+\nu)} \int_0^1 e^{-2izu}\, {}_2F_1(-n, 2\nu+n; \tfrac{1}{2}+\nu; 1-u)\, (1-u)^{\nu-\frac{1}{2}}\, u^{\nu-\frac{1}{2}}\, du.$$

Dieser Ausdruck geht mit Hilfe von (16) über in

$$\frac{i^n (2z)^\nu e^{iz}\, \Gamma(\frac{1}{2}+\nu)}{\sqrt{\pi}\, \Gamma(1-n)\, \Gamma(1+2\nu+n)}\, {}_2F_2(\tfrac{1}{2}+\nu, 1; 1-n, 1+2\nu+n; -2iz)$$

$$= \frac{i^n (2z)^\nu e^{iz}}{\sqrt{\pi}} \sum_{h=n}^{\infty} \frac{\Gamma(\frac{1}{2}+\nu+h)\, (-2iz)^h}{\Gamma(1-n+h)\, \Gamma(1+2\nu+n+h)}$$

$$= \frac{(2z)^{\nu+n} e^{iz}\, \Gamma(\frac{1}{2}+\nu+n)}{\sqrt{\pi}\, \Gamma(1+2\nu+2n)}\, {}_1F_1(\tfrac{1}{2}+\nu+n; 1+2\nu+2n; -2iz)$$

$$= \frac{z^{\nu+n} e^{iz}}{2^{\nu+n}\, \Gamma(1+\nu+n)}\, {}_1F_1(\tfrac{1}{2}+\nu+n; 1+2\nu+2n; -2iz).$$

Die rechte Seite dieser Beziehung ist gleich [31]) $J_{\nu+n}(z)$, womit (42) aus (16) abgeleitet ist.

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[28]) WATSON, [**18**], 50.

[29]) Ist $\nu = 0$, so nimmt man auch $n = 0$, also $\prod\limits_{h=0}^{n-1} (2\nu+h) = 1$.

[30]) APPELL–KAMPÉ DE FÉRIET, [**1**], 390.

[31]) WATSON, [**18**], 191.

§ 3. Ich werde jetzt die folgende Hilfsformel beweisen

$$\frac{\prod_{h=1}^{n}\Gamma(\lambda_h)}{\prod_{h=1}^{n-1}\Gamma(\mu_h)}\int_0^{(1+)} {}_nF_{n-1}\left(\begin{matrix}\lambda_1,\ldots,\lambda_n;\\ \mu_1,\ldots,\mu_{n-1};\,1/u\end{matrix}\right) u^{-s-1}\,du = 2\pi i\,\frac{\prod_{h=1}^{n}\Gamma(\lambda_h-s)}{\Gamma(1-s)\prod_{h=1}^{n-1}\Gamma(\mu_h-s)}; \tag{43}$$

hierbei wird

$n=1,2,3,\ldots,\ \lambda_h\neq 0,-1,-2,\ldots$ und $\Re(\lambda_h-s)>0\quad(h=1,\ldots,n)$

vorausgesetzt.

Ist $v>0$, $|\arg z|<\pi$ und $\lambda_h\neq 0,-1,-2,\ldots$, so gilt nämlich nach der BARNESschen Theorie der hypergeometrischen Funktionen

$$\frac{\prod_{h=1}^{n}\Gamma(\lambda_h)}{\prod_{h=1}^{n-1}\Gamma(\mu_h)}\,{}_nF_{n-1}\left(\begin{matrix}\lambda_1,\ldots,\lambda_n;\\ \mu_1,\ldots,\mu_{n-1};\,-v/z\end{matrix}\right)=\frac{1}{2\pi i}\int_{-\infty i+\tau}^{\infty i+\tau}\frac{\Gamma(s)\prod_{h=1}^{n}\Gamma(\lambda_h-s)}{\prod_{h=1}^{n-1}\Gamma(\mu_h-s)}(v/z)^{-s}\,ds; \tag{44}$$

hierin ist τ beliebig und der Integrationsweg lässt die Pole von $\Gamma(s)$ zur Linken und von $\prod_{h=1}^{n}\Gamma(\lambda_h-s)$ zur Rechten.

Ist $\operatorname*{Min}_{h=1,\ldots,n}\{\Re(\lambda_h)\}>0$, so kann man jede Gerade $\Re(s)=\tau$ mit $0<\tau<\operatorname*{Min}_{h=1,\ldots,n}\{\Re(\lambda_h)\}$ als Integrationsweg wählen. Ueberdies darf man dann auf (44) die MELLINsche Umkehrformel [32]) anwenden; das Resultat wird für $0<\Re(s)<\operatorname*{Min}_{h=1,\ldots,n}\{\Re(\lambda_h)\}$ und $|\arg z|<\pi$

$$\frac{\prod_{h=1}^{n}\Gamma(\lambda_h)}{\prod_{h=1}^{n-1}\Gamma(\mu_h)}\int_0^{\infty}{}_nF_{n-1}\left(\begin{matrix}\lambda_1,\ldots,\lambda_n;\\ \mu_1,\ldots,\mu_{n-1};\,-v/z\end{matrix}\right)v^{s-1}\,dv=\frac{z^s\,\Gamma(s)\prod_{h=1}^{n}\Gamma(\lambda_h-s)}{\prod_{h=1}^{n-1}\Gamma(\mu_h-s)} \tag{45}$$

Diese Beziehung gilt offenbar auch für $\arg z=\pm\pi$, wofern der Integrationsweg im Falle $\arg z=\pi$ den Punkt $v=-z$ durch einen oberhalb, im Falle $\arg z=-\pi$ aber durch einen unterhalb der reellen Achse liegenden Halbkreis vermeidet.

Setzt man nun $z=e^{\pi i}$ bezw. $z=e^{-\pi i}$ in (45) und subtrahiert man die so gefundenen Beziehungen, so erhält man

$$\frac{\prod_{h=1}^{n}\Gamma(\lambda_h)}{\prod_{h=1}^{n-1}\Gamma(\mu_h)}\int_{\infty}^{(1+)}{}_nF_{n-1}\left(\begin{matrix}\lambda_1,\ldots,\lambda_n;\\ \mu_1,\ldots,\mu_{n-1};\,v\end{matrix}\right)v^{s-1}\,dv$$

$$=\frac{(e^{-s\pi i}-e^{s\pi i})\,\Gamma(s)\prod_{h=1}^{n}\Gamma(\lambda_h-s)}{\prod_{h=1}^{n-1}\Gamma(\mu_h-s)}=-2\pi i\,\frac{\prod_{h=1}^{n}\Gamma(\lambda_h-s)}{\Gamma(1-s)\prod_{h=1}^{n-1}\Gamma(\mu_h-s)}.$$

32) COURANT–HILBERT, [3], 90—91.

Da der Punkt $v = 0$ nicht mehr auf dem Integrationswege liegt, gilt diese Formel für $\underset{h=1,\ldots,n}{\operatorname{Min}} \Re(\lambda_h - s) > 0$.

Ersetzt man nun noch v durch $1/u$, so findet man (43).

§ 4. Die in § 1 gefundenen Beziehungen (15) und (16) sind erweiterungsfähig. Für $n = 1, 2, 3, \ldots$ gilt nämlich [33])

$$\left.\begin{array}{l} {}_pF_q\begin{pmatrix} a_1, \ldots, a_p; \\ c_1, \ldots, c_q; z \end{pmatrix} = \dfrac{1}{2\pi i} \prod\limits_{h=1}^{n} \left\{ \dfrac{\Gamma(\alpha_h)\,\Gamma(1+\sigma_h-\alpha_1)}{\Gamma(\sigma_h)\,\Gamma(1+\alpha_h-\alpha_1)} \right\} \\ \times \displaystyle\int_0^{(+1)} {}_{p+n}F_{q+n}\begin{pmatrix} \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_p; \\ \sigma_1, \ldots, \sigma_n, c_1, \ldots, c_q; z u \end{pmatrix} \cdot {}_nF_{n-1}\begin{pmatrix} 1+\sigma_1-\alpha_1, \ldots, 1+\sigma_n-\alpha_1; \\ 1+\alpha_2-\alpha_1, \ldots, 1+\alpha_n-\alpha_1; 1/u \end{pmatrix} u^{\alpha_1-2}\, du; \end{array}\right\} (46)$$

hierin sind α_h und σ_h beliebig mit

$\Re(\sigma_h) > 0,\ \alpha_h \neq 0, -1, -2, \ldots,\ 1 + \sigma_h - \alpha_1 \neq 0, -1, -2, \ldots \quad (h = 1, \ldots, n)$.

Ferner hat man, gleichfalls für $n = 1, 2, 3, \ldots$, falls $\Re(\sigma_h) > 0$ und $1 + \sigma_h - \alpha_1 \neq 0, -1, -2, \ldots$ $(h = 1, \ldots, n)$ ist [34]),

$$\left.\begin{array}{l} 2\pi i \dfrac{\prod\limits_{h=1}^{n} \Gamma(\sigma_h)}{\prod\limits_{h=1}^{n} \Gamma(\alpha_h)}\, {}_{p+n}F_{q+n}\begin{pmatrix} a_1, \ldots, a_p, \sigma_1, \ldots, \sigma_n; \\ c_1, \ldots, c_q, \alpha_1, \ldots, \alpha_n; z \end{pmatrix} = \dfrac{\prod\limits_{h=1}^{n} \Gamma(1+\sigma_h-\alpha_1)}{\prod\limits_{h=1}^{n} \Gamma(1+\alpha_h-\alpha_1)} \\ \times \displaystyle\int_0^{(1+)} {}_pF_q\begin{pmatrix} a_1, \ldots, a_p; \\ c_1, \ldots, c_q; z u \end{pmatrix} \cdot {}_nF_{n-1}\begin{pmatrix} 1+\sigma_1-\alpha_1, \ldots, 1+\sigma_n-\alpha_1; \\ 1+\alpha_2-\alpha_1, \ldots, 1+\alpha_n-\alpha_1; 1/u \end{pmatrix} u^{\alpha_1-2}\, du. \end{array}\right\} (47)$$

Der Beweis von (46) und (47) geht durch Entwicklung der Funktionen $F(zu)$ und gliedweise Integration, wobei die aus (43) folgende Formel

$$\frac{\prod\limits_{h=1}^{n} \Gamma(1+\sigma_h-\alpha_1)}{\prod\limits_{h=1}^{n} \Gamma(1+\alpha_h-\alpha_1)} \cdot \int_0^{(1+)} {}_nF_{n-1}\begin{pmatrix} 1+\sigma_1-\alpha_1, \ldots, 1+\sigma_n-\alpha_1; \\ 1+\alpha_2-\alpha_1, \ldots, 1+\alpha_n-\alpha_1; 1/u \end{pmatrix} u^{\alpha_1+l-2}\, du = 2\pi i \frac{\prod\limits_{h=1}^{n} \Gamma(\sigma_h+l)}{\prod\limits_{h=1}^{n} \Gamma(\alpha_h+l)}$$

angewendet werden muss.

Es wird sich zeigen, dass (46) und (47) mit $n = 2$ nach einiger Transformation in (15) bezw. (16) übergehen. Ich werde aber erst die Spezialfälle mit $n = 1$ untersuchen. Wegen

$${}_1F_0(1+\sigma-\alpha; 1/u) = (1-1/u)^{\alpha-\sigma-1} = (u-1)^{\alpha-\sigma-1}\, u^{\sigma+1-\alpha}$$

[33]) Ist $p = q + 1$, so ist die z-Ebene von $z = 1$ bis $z = \infty$ aufgeschnitten.
[34]) Man vergl. Fussnote [7]).

liefert (46) mit $n=1$, $a_1=a$ und $\sigma_1=\sigma$

$$\left.\begin{aligned} &{}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;z)=\frac{1}{2\pi i}\frac{\Gamma(\alpha)\,\Gamma(1+\sigma-\alpha)}{\Gamma(\sigma)} \\ &\times\int_0^{(1+)}{}_{p+1}F_{q+1}(\alpha,a_1,\ldots,a_p;\sigma,c_1,\ldots,c_q;zu)\,(u-1)^{\alpha-\sigma-1}\,u^{\sigma-1}\,du. \end{aligned}\right\}\quad .\ .\ (48)$$

Der entsprechende Spezialfall von (47) ergibt

$$\left.\begin{aligned} &2\pi i\,\frac{\Gamma(\sigma)}{\Gamma(\alpha)}\,{}_{p+1}F_{q+1}(a_1,\ldots,a_p,\sigma;c_1,\ldots,c_q,\alpha;z) \\ &=\Gamma(1+\sigma-\alpha)\int_0^{(1+)}{}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;zu)\,(u-1)^{\alpha-\sigma-1}\,u^{\sigma-1}\,du. \end{aligned}\right\}\quad (49)$$

Die Relationen (48) und (49) waren schon bekannt [35].

Ich nehme nun $n=2$ in (46), ich setze $a_1=\alpha$, $a_2=\beta$, $\sigma_1=\sigma$ und $\sigma_2=\tau$ und benutze die Beziehung [36])

$$\frac{\Gamma(1+\sigma-\alpha)\,\Gamma(1+\tau-\alpha)}{\Gamma(1+\beta-\alpha)}\,{}_2F_1\binom{1+\sigma-\alpha,\,1+\tau-\alpha;}{1+\beta-\alpha;\,1/u}$$

$$=\frac{\Gamma(1+\sigma-\alpha)\,\Gamma(1+\tau-\alpha)\,\Gamma(\alpha+\beta-\sigma-\tau-1)}{\Gamma(\beta-\sigma)\,\Gamma(\beta-\tau)}\,u^{1+\sigma-\alpha}\,{}_2F_1\binom{1+\sigma-\alpha,\,1+\sigma-\beta;}{2+\sigma+\tau-\alpha-\beta;\,1-u}$$

$$+\,\Gamma(1+\sigma+\tau-\alpha-\beta)\,u^{1+\sigma-\alpha}\,(u-1)^{\alpha+\beta-\sigma-\tau-1}\,{}_2F_1\binom{\alpha-\tau,\,\beta-\tau;}{\alpha+\beta-\sigma-\tau;\,1-u};$$

wegen [37])

$$\int_0^{(1+)}{}_{p+2}F_{q+2}(zu)\cdot{}_2F_1\binom{1+\sigma-\alpha,\,1+\sigma-\beta;}{2+\sigma+\tau-\alpha-\beta;\,1-u}\,u^{\sigma-1}\,du=0$$

geht (46) mit $n=2$ also in

$$\left.\begin{aligned} &{}_pF_q\binom{a_1,\ldots,a_p;}{c_1,\ldots,c_q;\,z}=\frac{\Gamma(1+\sigma+\tau-\alpha-\beta)\,\Gamma(\alpha)\,\Gamma(\beta)}{2\pi i\,\Gamma(\sigma)\,\Gamma(\tau)}\int_0^{(1+)}{}_{p+2}F_{q+2}\binom{\alpha,\beta,a_1,\ldots,a_p;}{\sigma,\tau,c_1,\ldots,c_q;\,zu} \\ &\qquad\times{}_2F_1(\alpha-\tau,\beta-\tau;\alpha+\beta-\sigma-\tau;1-u)\,(u-1)^{\alpha+\beta-\sigma-\tau-1}\,u^{\sigma-1}\,du \end{aligned}\right\}(50)$$

über. Ist $\Re(\alpha+\beta-\sigma-\tau)>0$, so kann diese Relation auf dem üblichen Wege auf (15) zurückgeführt werden, womit dann (15) aus (46) abgeleitet ist.

Auf analoge Weise folgt (16) aus (47) mit $n=2$.

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35) Man vergl. (12); siehe auch ERDÉLYI, [6].

36) BARNES, [2], 152, Formel (IX).

37) Der Integrand ist analytisch innerhalb des Integrationsweges.

Ist $n \geqq 2$, so gilt für die auf der rechten Seite von (46) vorkommende Funktion ${}_nF_{n-1}(1/u)$ in der Umgebung der singulären Stelle $u=1$ die Entwicklung

$$\left.\begin{aligned}&\frac{\prod\limits_{h=1}^{n} \Gamma(1+\sigma_h-\alpha_1)}{\prod\limits_{h=1}^{n} \Gamma(1+\alpha_h-\alpha_1)}\,{}_nF_{n-1}\left(\begin{matrix}1+\sigma_1-\alpha_1,\ldots,1+\sigma_n-\alpha_1;\\ 1+\alpha_2-\alpha_1,\ldots,1+\alpha_n-\alpha_1;1/u\end{matrix}\right)\\ &=u^{\lambda}\,\Phi(1-u)+\Gamma(1+\sigma_1+\ldots+\sigma_n-\alpha_1-\ldots-\alpha_n)\,u^{1+\sigma_1-\alpha_1}(u-1)^{\alpha_1+\ldots+\alpha_n-\sigma_1-\ldots-\sigma_n-1}\,\Psi(1-u);\end{aligned}\right\}\tag{51}$$

die Potenzreihen $\Phi(v)$ und $\Psi(v)$ sind für $n>2$ keine hypergeometrische Reihen, sondern Reihen mit sehr kompliziert gebauten Koeffizienten, die ich hier nicht angeben werde [38]).

Setzt man nun die Entwicklung (51) in (46) ein, so findet man, gleich wie soeben für $n=2$,

$$ {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p;\\ c_1,\ldots,c_q;z\end{matrix}\right)=\frac{\Gamma(1+\sigma_1+\ldots+\sigma_n-\alpha_1-\ldots-\alpha_n)\prod\limits_{h=1}^{n}\Gamma(\alpha_h)}{2\pi i\prod\limits_{h=1}^{n}\Gamma(\sigma_h)}$$

$$\times\int_0^{(1+)} {}_{p+n}F_{q+n}\left(\begin{matrix}\alpha_1,\ldots,\alpha_n,a_1,\ldots,a_p;\\ \sigma_1,\ldots,\sigma_n,c_1,\ldots,c_q;zu\end{matrix}\right)\Psi(1-u)\,(u-1)^{\alpha_1+\ldots+\alpha_n-\sigma_1-\ldots-\sigma_n-1}\,u^{\sigma_1-1}\,du;$$

diese Beziehung ist eine Erweiterung von (50).

§ 5. Die in den Paragraphen 1 und 4 bewiesenen Relationen erinnern in gewissen Hinsichten an die Beziehungen, die ich neuerdings in meiner Arbeit über die KUMMERsche Funktion [39]) abgeleitet habe. Das dort gefundene Resultat kann auf verschiedene Weisen erweitert werden. Auf dieselbe Weise wie damals Formel (8) beweist man z.B. die etwas allgemeinere, für $\Re(a)>0$ gültige Formel

$$\frac{\Gamma(a)}{\Gamma(b)}\,{}_1F_1(a;b;z)=c\int_{\infty}^{(0+,z+)} {}_3F_3\left(\begin{matrix}\alpha,\beta,a+\lambda;\\ \sigma,\tau,b+\lambda;u\end{matrix}\right)\cdot{}_3F_2\left(\begin{matrix}1-\lambda,\sigma-\lambda,\tau-\lambda;\\ \alpha-\lambda,\beta-\lambda;z/u\end{matrix}\right)u^{\lambda-1}\,du,$$

worin

$$c=\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(a+\lambda)\,\Gamma(1-\lambda)\,\Gamma(\sigma-\lambda)\,\Gamma(\tau-\lambda)}{2\pi i\,\Gamma(\sigma)\,\Gamma(\tau)\,\Gamma(b+\lambda)\,\Gamma(\alpha-\lambda)\,\Gamma(\beta-\lambda)}.$$

Hierin sind α,β,σ,τ und λ beliebig mit

$\Re(\alpha-\lambda)>0$, $\Re(\beta-\lambda)>0$, $\alpha\neq 0,-1,-2,\ldots$, $\beta\neq 0,-1,-2,\ldots$, $a+\lambda\neq 0,-1,-2,\ldots$, $\lambda\neq 1,2,3,\ldots$, $\sigma-\lambda\neq 0,-1,-2,\ldots$, $\tau-\lambda\neq 0,-1,-2,\ldots$.

38) $\Psi(0)=1$; für die genaue Gestalt der Entwicklung (51) siehe man WINKLER, [19]. Der Fall mit $n=3$ ist von DARLING, [4], untersucht worden.

39) MEIJER, [16].

LITERATURVERZEICHNIS.

P. APPELL et J. KAMPÉ DE FÉRIET.

1. Fonctions hypergéométriques et hypersphériques. Polynomes d'HERMITE (1926).

E. W. BARNES.

2. A new development of the theory of the hypergeometric functions. Proc. London Math. Soc., (2) **6**, 141—177 (1908).

R. COURANT und D. HILBERT.

3. Methoden der mathematischen Physik. I (1924).

H. B. C. DARLING.

4. On the differential equation satisfied by the hypergeometric series of the second order. Journ. London Math. Soc., **10**, 63—70 (1935).

A. ERDÉLYI.

5. Der Zusammenhang zwischen verschiedenen Integraldarstellungen hypergeometrischer Funktionen. Quarterly Journ. of Math. (Oxford series), **8**, 200—213 (1937).
6. Integraldarstellungen hypergeometrischer Funktionen. Ibidem, **8**, 267—277 (1937).

C. F. GAUSS.

7. Disquisitiones generales circa seriem infinitam .... Werke, **3**, 125—162 (1866).

E. W. HOBSON.

8. The theory of spherical and ellipsoidal harmonics (1931).

E. E. KUMMER.

9. Ueber die hypergeometrische Reihe .... Journ. für die reine und angewandte Math., **15**, 39—83 und 127—172 (1836).

T. M. MACROBERT.

10. The MEHLER–DIRICHLET integral and some other LEGENDRE function formulae. Philosophical Magazine, (7) **14**, 632—656 (1932).

C. S. MEIJER.

11. Neue Integraldarstellungen aus der Theorie der WHITTAKERschen und HANKELschen Funktionen. Math. Annalen, **112**, 469—489 (1936).
12. Einige Integraldarstellungen aus der Theorie der BESSELschen und WHITTAKERschen Funktionen. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 394—403 und 519—527 (1936).
13. Ueber Produkte von WHITTAKERschen Funktionen. Ibidem, **40**, 133—141 und 259—263 (1937).
14. Einige Integraldarstellungen für die LOMMELsche Funktion $S_{\mu,\nu}(z)$. Ibidem, **41**, 151—154 (1938).
15. Note über das Produkt $M_{k,m}(z)\,M_{-k,m}(z)$. Ibidem, **41**, 275—277 (1938).
16. Ueber die KUMMERsche Funktion ${}_1F_1\,(a; b; z)$. Ibidem, **41**, 1108—1114 (1938).

P. SZYMAŃSKI.

17. On the integral representations of the LOMMEL functions. Proc. London Math. Soc., (2) **40**, 71—82 (1936).

G. N. WATSON.

18. A treatise on the theory of BESSEL functions (1922).

E. WINKLER.

19. Ueber die hypergeometrische Differentialgleichung n^{ter} Ordnung mit zwei endlichen singulären Punkten. Dissertation München (1931).

Neurology. — *On a new form of experimental position-nystagmus in the rabbit and its clinical value.* By A. BIEMOND, chef-de-clinique of the Neurological Clinic, Wilhelmina Gasthuis. (From the Neurological Clinic (Director Prof. B. BROUWER) and the Otological Clinic (Director Prof. A. DE KLEYN), University of Amsterdam.) (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of March 25, 1939.)

In 1913 ROTHFELD described a regularly appearing position-nystagmus in rabbits intoxicated with alcohol. ROTHFELD thought that this nystagmus was a consequence of the suppression of the compensatory position of the eyes. DE KLEYN and VERSTEEGH, however, in 1930 demonstrated that the relation seen by ROTHFELD was present only in cases of more severe intoxication while with lighter degrees of intoxication the position-nystagmus and compensatory eye-positions appeared simultaneously.

Even before this experimental demonstration, position-nystagmus had been observed in man by OPPENHEIM who in 1912 described a patient with a tumour in the posterior fossa of the skull. This patient showed a nystagmus only when the body was in a lateral position. The next clinical observations came from BÁRÁNY who in 1913 and 1921 described patients who complained of dizziness and showed nystagmus only when the head occupied a definite position. An analogous observation was made by VOSS (1921) who tried to show that only tonic labyrinthine reflexes were responsible for the position-nystagmus seen in his patient. In 1922 this interpretation was criticized by DE KLEYN and VERSTEEGH who made it clear that in VOSS' patient also neck reflexes could be active. These authors then reported a case of position-nystagmus in man is which by exact testing it could be shown that here, at least, the nystagmus occurred only by changing the position of the head with regard to the trunk and not merely by changing the position of the head in space.

Later on, position-nystagmus was repeatedly seen by other clinicians in cases of tumour of the posterior fossa. Casual observations of this kind were published by BUYS, MARTIN and VAN BOGAERT (1926), WINTHER (1930), GUILLAIN (1935) and others.

In a systematic way this problem was studied in 1931 by NYLÉN who investigated 150 cases of brain tumour for the precense of position-nystagmus. This symptom appeared to have a great diagnostic significance for cases of tumours of the posterior fossa of which 50—60 % showed this sign. NYLÉN assumed that the cause of position-nystagmus in cases of

tumour lay in the changing pressure by the tumour on the vestibular nuclei, when the position of head and trunk was altered.

It is clear however that the "pressure"-theory cannot hold in those cases of position-nystagmus in man where there is no tumour present. There is a further difficulty that position-nystagmus has been seen in cases of peripheral labyrinthine disease as well as in patients with perfectly normal peripheral apparatus. One speaks therefore of peripheral and central position-nystagmus. And, finally, it was shown that tonic labyrinthine as well as neck reflexes can play a rôle in this mechanism. Therefore one must say that for the time being we are groping in the dark for the cause and the significance of this peculiar symptom.

The following recent clinical observations made in the otological and neurological university clinics of the Wilhelmina Gasthuis in Amsterdam, led to a renewed study of this problem.

Case 1. A woman of 41 years was admitted to the otological clinic on September 22, 1938 complaining of severe dizziness and vomiting. The patient reported that these symptoms started some weeks earlier, more or less in association with a common febrile condition. The dizziness occurred especially when the patient was lying in the right lateral position (R.L.P.). In fact it appeared from the clinical investigation that a distinct horizontal nystagmus to the right was present with the patient in this same position. It was not present with the patient in the recumbent and the left lateral position (L.L.P.). The nystagmus was even more marked when the patient lay first for some moments in the L.L.P. and then was changed into the R.L.P. The hearing was intact on both sides. No tinnitus. The vestibular reactions (caloric and rotatory) were normal and equal on both sides. The reactions with the basculing table (RADEMAKER and GARCIN) were normal. On October 10th the patient was transferred to the neurological clinic to complete the examination. From this it appeared that the patient had an hypaesthetic zone in a small area covering a part of the right side of the neck, the right external ear and the adjoining part of the occiput. The boundaries of this zone agreed approximately with the dermatomic representation of C2. Furthermore, it was stated that ERB's point in the right supraclaviculor fossa was painful upon pressure and that the patient had been afflicted since 1934 by a from time to time recidivating brachial neuritis of the right arm. For this affection she had been treated some years ago in a neurological out-patient department. Other neurological disturbances were not stated. All reflexes were normal. X-rays of the skull and vertebral column gave normal pictures. Also, by lumbar puncture no abnormalities were found. The investigations of the internal organs, blood and urine were negative.

The complaints of dizziness and the finding of hypaesthesia in C2 gradually diminished without treatment. After some weeks the symptom of position-nystagmus could no longer be elicited. The patient remained a little bit dizzy for a rather long time. Nevertheless, on January 14th she was discharged without objective or subjective symptoms.

Case 2. A woman of 37 years, who shows a typical achondroplasia, was admitted on September 7, 1938 to the otological clinic, complaining of dizziness existing for some weeks and especially occurring in the L.L.P. From the clinical examination it appeared that a distinct horizontal position-

nystagmus was present with the patient in the L.L.P. Also here the symptom was more marked when the patient was placed first in the R.L.P. and then changed to the L.L.P. The hearing was good on the right side but greatly diminished on the left. There was a dry perforation of the left tympanum. There was no discharge. The vestibular reactivity (only rotatory) was normal and equal bilaterally. Also the balancing reactions were normal. On September 19 the patient was transferred to the neurological clinic. Here it was found that there was a distinct left-sided brachial neuritis with pressure pain on ERB's point, on the left side of the neck and also along the peripheral nerve trunks. There existed paraesthesias in the fingertips of the left hand but no objective disturbances of sensibility or motility. The patient told us, that she had had these symptoms for a long time, long before the appearance of the dizziness. The tendency to dizziness from the assumption of a certain position of the body and the sensitiveness of the left arm are still present. The position-nystagmus, however, is gradually diminishing. Diathermy of the left shoulder and arm has been given with some beneficial effect. The remaining neurological and internal investigation gave no positive findings. The X-rays of the skull, vertebral column and long bones of the extremities showed the typical abnormalities of achondroplasia. No fluid was obtained by lumbar puncture.

Case 3. A woman of 44 years was admitted on December 14, 1938, to the neurological clinic. Three weeks previously pain appeared low in the right side of the neck. Some days later the pain extended into the right shoulder and in 10 days more the right arm and hand, also, were involved. The pain appeared subjectively to stream from the shoulder to the fingertips. Four days prior to admission the patient experienced dizziness (with typical rotatory sensations). She could find relief from the dizziness only when she turned her head to the right or was lying on the right side.

Clinical examination revealed a distinct spontaneous horizontal nystagmus to the right, even present in the recumbent position, diminishing and often disappearing in the R.L.P. or on turning the head to the right and becoming very marked in the L.L.P. or on turning the head to the left. The hearing was normal on both sides. The vestibular reactivity (caloric and rotatory) was equally normal (otological clinic). The balancing reactions were not examined. From the neurological examination it appeared further, that the right cervico-brachial plexus was very painful to pressure; the triceps reflex was absent on the right, the strength of the right triceps muscle was diminished. The second and third finger of the right hand were hypaesthetic. Other disturbances were not noted. The X-ray examination of the skull and cervical vertebrae revealed no abnormalities. Both lumbar and suboccipital punctures were done and gave normal findings. The patient was treated exclusively for the neuritic symptoms of right arm and neck (dry heat, salicylates, diathermy). As these neuritic symptoms improved, there was a parallel reduction of the dizziness and position-nystagmus. When the patient was discharged from the clinic (February 4, 1939) she was entirely free from complaints. A follow-up record of March 23 revealed that she remained in good health.

Thus we have described 3 patients, all of whom had in the course of a neuralgia (neuritis or radiculitis) cervico-brachialis complaints of dizziness occurring with a definite position of head and body and all of whom showed on examination a position-nystagmus in these same positions. In all 3 patients the vestibular reactivity was normal; in 2 of the 3 cases the

hearing was intact, also. The position-nystagmus pointed always with the quick component to the side of the affected arm (in cases 1 and 3 to the right, in case 2 to the left). It was elicited in cases 1 and 2 by lying on the side of the affected extremity. In case 3 just the opposite relation obtained; here the nystagmus was most marked on turning the head (or head and trunk) to the unaffected side. In 2 cases the dizziness and position-nystagmus disappeared almost simultaneously with the cure of the neuritic symptoms. In one case both affections still remain.

The above described observations raised the idea that there existed a causative relationship between the cervico-brachial affection and the occurrence of dizziness and position-nystagmus. I therefore tried to establish this relationship also by experimental investigation. In order to do this the second cervical root was cut at the point of emergence from the intervertebral foramen in a series of rabbits, operated upon in the laboratory of Prof. A. DE KLEYN. The operations were made under aether-narcosis. In most cases we succeeded easily, after displacing the neck muscles lateralward from the middle, in finding and cutting the second cervical root in the intervertebral foramen. In a few cases there occurred a severe venous bleeding from the deeper regions resulting in the death of the animal. In the large majority of cases the operation was easily accomplished. In 6 rabbits the right, in 4 the left second cervical root was cut. In 8 of these 10 cases (5 on the right, 3 on the left) there occurred, a few minutes after the cutting (while the animal remained on the operating table and narcosis was permitted to become light), *a typical position-nystagmus*. Whereas in the normal, upright position of the head no trace of nystagmus was to be seen, it appeared immediately after turning the head (the trunk remaining fixed) to the right or to the left. This nystagmus was, in every case, in the upper eye, at first, a few minutes after cutting, of a rotatory character; later on more and more purely horizontal. The quick phase of the rotation turned the upper pole of the eye in a temporal direction. The horizontal nystagmus which followed pointed with the quick phase forward (and somewhat downward). The lower eye presented, at the same time (equal on right or left turning of the head) a horizontal nystagmus with the quick phase backward (and somewhat upward). The nystagmus was thus of an associated character. In only one experiment was a nystagmus of dissociated character observed. In all cases a compensatory eye-position was normally present after turning of the head. The described nystagmus was observed in the same manner when the animal, after closing of the wound, removal from the operating table and disappearing of the narcosis, was placed as a whole in the right or left lateral position so that the position of the head with relation to the trunk remained unchanged. On rotation of the animal in different positions around a bitemporal axis of the head, no nystagmus occurred. This test was performed in only one experiment, but it must be pointed out again that in all of the experiments there was no nystagmus in the normal position of the head and trunk in space.

The position-nystagmus could be elicited only during a restricted time (an average of $\frac{1}{2}$ to $\frac{3}{4}$ of an hour) after the operation. In this period the intensity gradually diminished, the nystagmus became slower and the excursions followed each other at greater intervals. By giving a sensory stimulus (f.i. pinching the back of the neck) we were able repeatedly to reinforce a nystagmus which was nearly extinct. We got the impression that the position-nystagmus in the lateral position opposite to the side of cutting usually persisted somewhat longer than in the ipsilateral.

Without exception, the animal, after the operation, held his head in a normal position. After recovering from the narcosis the animal could walk in a normal manner.

In 3 further experiments the second cervical root (in this case the right) was left intact and was stimulated galvanically with a bipolar electrode on the same spot where in the former experiments the cutting was performed. In 2 of these 3 experiments a distinct position-nystagmus occurred, in which the form and direction were similar in the upper and lower eyes to those described above in the cutting experiments.

In one experiment the right and left second cervical roots were cut simultaneously. In this case, also, a distinct position nystagmus occurred, in all ways similar to that described above.

Several times we examined normal rabbits under narcosis for the presence of position-nystagmus. It was never seen, so that narcosis cannot play a significant rôle in its production. The investigation remained restricted to cutting (or stimulation) of the *second* (right or left) cervical root. An operation on the first cervical root was not done because in this case a complicating lesion of the floor of the fourth ventricle by bleeding could not be excluded with sufficient certainty. The third cervical root which ramifies directly after its emergence from the intervertebral foramen, can be completely cut only by opening the vertebral canal. This operation presents great technical difficulties such as severe venous hemorrhage which is usually fatal. In the few experiments of this type we have done we have not succeeded in cutting this root. It is, of course, desirable to complete the experiments by further attacks on the third and lower cervical roots.

Thus it can be concluded with sufficient certainty that cutting (or stimulation) of the second cervical root in rabbits elicits a typical position-nystagmus.

From these experimental findings in rabbits and the above mentioned clinical observations, it can be considered probable, that also in man a process of the cervical roots can produce position-nystagmus and dizziness.

Apart from position-nystagmus caused by brain tumours localized in the posterior fossa, if we are correct, *there must be reserved a place in the clinical symptomatology for position-nystagmus of peripheral (cervical) origin.*

The combination of neuritis (or radiculitis) cervico-brachialis with

dizziness and position-nystagmus thus receives its own nosological significance.

Many aspects of this problem remain unsolved, however. A careful anatomical control of the described experiments has to be done. In 2 of our experimental animals the cervical cord and brainstem will be cut serially and stained by the MARCHI-method. Perhaps we shall thus get an insight into the pathways that a stimulus from the second cervical root must follow to produce an irritation of the vestibular nuclei. Our present anatomical knowledge in this respect is insufficient. It is possible that the position-nystagmus in our clinical cases was a manifestation of an unilateral irritation of the vestibular nucleus (nuclei). Noteworthy is it in this connection that the nystagmus in all 3 cases pointed with the quick component to the side of the affected arm. The lateral position in which the nystagmus occurred was, however, in the first and second cases directed towards the affected arm, in the third case just the opposite. In rabbits both lateral positions gave position nystagmus regardless of the side operated upon. To what extent here, also tonic neck reflexes are active, must further be investigated.

Certainly it must also be examined in how far cervical root irritation plays a rôle in the nystagmus and position-nystagmus described in certain cases of affection strictly limited to the spinal cord.

BIBLIOGRAPHY.

1. R. BáRáNY, Dauernde Veränderung der spontanen Nystagmus bei Veränderung der Kopflage. Monatschr. f. Ohrheilk., Jahrgang 47 (1913).
2. R. BáRáNY, Diagnose von Krankheitserscheinungen im Bereiche des Otolithenapparates. Acta Oto-Laryngologica, vol. II (1921).
3. BUYS, MARTIN et VAN BOGAERT, Gliome épendymaire du Cervelet. Journal de Neur. et de Psych., vol. XXV (1926).
4. G. GUILLAIN, Etudes neurologiques, VI Série, p. 71 (1935). Masson, Paris.
5. A. DE KLEYN und C. VERSTEEGH, Schwindelanfälle und Nystagmus bei einer bestimmten Lage des Kopfes. Acta Oto-Laryngologica, vol. VI (1922).
6. A. DE KLEYN und C. VERSTEEGH, Experimentelle Untersuchungen über der sogenannten Lagenystagmus während akuter Alcoholvergiftung beim Kaninchen. Acta Oto-Laryngologica, vol. XIV (1930).
7. C. O. NYLÉN, A clinical study on positional nystagmus in cases of brain tumour. Acta Oto-Laryngologica, suppl. XV (1931).
8. H. OPPENHEIM, Ueber einen Fall operativ behandelter Kleinhirngeschwulst mit Heilerfolg. Berliner klinische Wochenschrift, Nr. 50 (1912).
9. J. ROTHFELD, Ueber den Einflusz acuter und chronischer Alcoholvergiftung auf die vestibulären Reaktionen. Arb. aus dem neurol. Institut der Wiener Universität, Bd. XX (1913).
10. O. VOSS, Erkrankungen des Otolithenapparates und die Methode zu deren Feststellung. Verhandl. d. Gesellsch. D. Hals-, Nasen- und Ohrenärzte, I. Versammlung, Mai 1921.